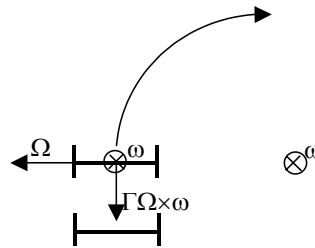


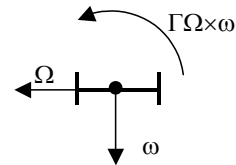
Simulation : [http://www.ulb.ac.be/polytech/smana/2008\\_2009/seance11.htm](http://www.ulb.ac.be/polytech/smana/2008_2009/seance11.htm)

1. Gyrostat :  $\Gamma \bar{\Omega} \times \bar{\omega}$   
les roues droites de la voiture ont tendance à décoller.

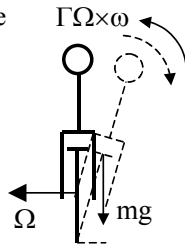
Vue de haut



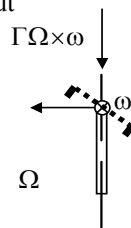
Vue de derrière



- 2.a Vue de derrière



Vue de haut



Si le cycliste se penche vers la droite, la force de pesanteur génère un moment qui a tendance à faire basculer le cycliste. Pour ne pas tomber, il doit générer un moment opposé qui le ramène en position verticale. En tournant son guidon vers la droite (du côté où il penche), il crée un effet gyroscopique qui le ramène dans sa position verticale.

- 2.b Roue = 60 cm de diamètre ;

$$\Gamma = 0,6 \cdot (0,3)^2$$

$$v = \Omega R \Rightarrow 8 = \Omega \cdot 0,3 \Rightarrow \Omega = 8/0,3 \text{ rad/s}$$

$$\bar{m}_{e,C} = \bar{CG} \times m \bar{g} = 0,03 \cdot 80,9 \cdot 81$$

$$\Gamma \bar{\Omega} \times \bar{\omega} = 0,6 \cdot (0,3)^2 \frac{8}{0,3} \omega$$

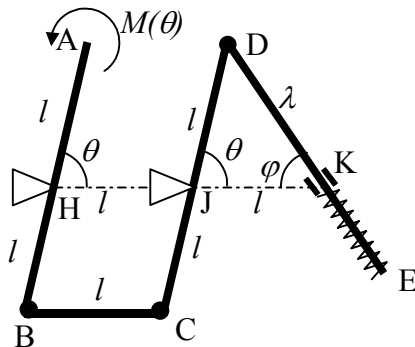
Pour retrouver l'équilibre :

$$\frac{d}{dt} \bar{M}_C = \bar{m}_{e,C} + \Gamma \bar{\Omega} \times \bar{\omega}$$

$$\Rightarrow 0 = 0,03 \cdot 80,9 \cdot 81 - 0,6 \cdot (0,3)^2 \frac{8}{0,3} \omega$$

$$\omega = 16,35 \text{ rad/s}$$

- 3.1

3 Paramètre de position :  $\theta, \varphi, \lambda$ 

$$2 \text{ relations : } \begin{cases} \varphi = \frac{\pi}{2} - \frac{\theta}{2} \Rightarrow \dot{\varphi} = -\frac{\dot{\theta}}{2} \\ \lambda = 2L \sin \frac{\theta}{2} \text{ ou } L \sin \theta = \lambda \sin \varphi \end{cases} \Rightarrow 1 \text{ ddl.}$$

Force : 2 réactions en H, 2 réactions en J, 1 réaction en K perpendiculaire à DE, Force de rappel en E, poids des 4 tiges. Equation de mouvement : Lagrange.

Les réactions en H, J et K ne travaillent pas car les points sont fixes ou le déplacement se fait perpendiculaire aux forces.

Le couple extérieur appliqué en A ne dérive pas d'un potentiel  $\Rightarrow$  Application du théorème de Lagrange généralisé.

$$T = \sum_{i=1}^4 \left( \frac{1}{2} m v_{G_i}^2 + \frac{1}{2} \bar{\omega} \cdot \bar{I}_{G_i} \cdot \bar{\omega} \right)$$

$$T = \left( \frac{1}{2} 2m \frac{(2l)^2}{12} \dot{\theta}^2 \right)_1 + \left( \frac{m}{2} (l\dot{\theta})^2 \right)_2 + \left( \frac{1}{2} 2m \frac{(2l)^2}{12} \dot{\theta}^2 \right)_3 + \left( \frac{2m}{2} \frac{5}{4} l^2 \dot{\theta}^2 - \frac{2m}{2} l^2 \dot{\theta}^2 \sin \frac{\theta}{2} + \frac{1}{2} 2m \frac{(2l)^2}{12} \frac{\dot{\theta}^2}{4} \right)_4$$

$$\Rightarrow T = \left( \frac{5}{2} - \sin \frac{\theta}{2} \right) m l^2 \dot{\theta}^2$$

$$\text{avec } \begin{cases} \overline{JG_4} = (L \cos \theta + L \cos \varphi) \bar{I}_x + (L \sin \theta - L \sin \varphi) \bar{I}_y \\ \bar{v}_{G_4} = \bar{v}_D + (-\dot{\varphi} \bar{I}_z) \times \overline{DG_4} = L \dot{\theta} (-\sin \theta \bar{I}_x + \cos \theta \bar{I}_y) + \frac{\dot{\theta}}{2} L \left( \cos \frac{\theta}{2} \bar{I}_x + \sin \frac{\theta}{2} \bar{I}_y \right) = \frac{d(\overline{JG_4})}{dt} \\ v_{G_4}^2 = L^2 \dot{\theta}^2 + \frac{\dot{\theta}^2}{4} L^2 + L^2 \dot{\theta}^2 \left( -\sin \theta \cos \frac{\theta}{2} + \cos \theta \sin \frac{\theta}{2} \right) = \frac{5}{4} L^2 \dot{\theta}^2 + L^2 \dot{\theta}^2 \sin \left( -\frac{\theta}{2} \right) \end{cases}$$

$$\boxed{Q_\theta = \sum \bar{F}_h \frac{\partial \bar{\varphi}_h}{\partial q_\theta}} = -mg \bar{l}_y \cdot \frac{\partial \overline{OG_2}}{\partial \theta} - 2mg \bar{l}_y \cdot \frac{\partial \overline{JG_4}}{\partial \theta} + \bar{F}_r \cdot \frac{\partial \overline{JE}}{\partial \theta} + M(\theta)$$

$$\bar{F}_r = -k(L - L_0) \bar{l}_{DE} = k2L \sin \frac{\theta}{2} \bar{l}_{DE} = k2L \sin \frac{\theta}{2} \left( \sin \frac{\theta}{2} \bar{l}_x - \cos \frac{\theta}{2} \bar{l}_y \right)$$

$$\text{avec } \begin{cases} \overline{OG_2} = \left( -L \cos \theta + \frac{L}{2} \right) \bar{l}_x - L \sin \theta \bar{l}_y \Rightarrow \delta \overline{OG_2} = \left( L \sin \theta \bar{l}_x - L \cos \theta \bar{l}_y \right) \delta \theta = \frac{\partial \overline{OG_2}}{\partial \theta} \delta \theta \\ \overline{JG_4} \Rightarrow \delta \overline{JG_4} = \left( \left( -L \sin \theta + \frac{1}{2} L \cos \frac{\theta}{2} \right) \bar{l}_x + \left( L \cos \theta + \frac{1}{2} L \sin \frac{\theta}{2} \right) \bar{l}_y \right) \delta \theta = \frac{\partial \overline{JG_4}}{\partial \theta} \delta \theta \\ \overline{JE} = \left( L \cos \theta + 2L \cos \varphi \right) \bar{l}_x + \left( L \sin \theta - 2L \sin \varphi \right) \bar{l}_y = L \left( \cos \theta + 2 \sin \frac{\theta}{2} \right) \bar{l}_x + L \left( \sin \theta - 2 \cos \frac{\theta}{2} \right) \bar{l}_y \\ \Rightarrow \delta \overline{JE} = \left( L \left( -\sin \theta + \cos \frac{\theta}{2} \right) \bar{l}_x + L \left( \cos \theta + \sin \frac{\theta}{2} \right) \bar{l}_y \right) \delta \theta = \frac{\partial \overline{JE}}{\partial \theta} \delta \theta \end{cases}$$

Autre méthode, par le calcul du potentiel :

$$\boxed{Q_\theta = Q_{\theta(F \text{ dérivant de } V)} + Q_{\theta(F \text{ ne dérivant de } V)}} \text{ avec } Q_{\theta(F \text{ ne dérivant de } V)} = M(\theta)$$

$$Q_{\theta(F \text{ dérivant de } V)} = -\frac{\partial V}{\partial \theta} \text{ avec } V = \frac{k}{2} (L - L_0)^2 - mgL \sin \theta + 2mg \left( L \sin \theta - L \sin \varphi \right)$$

$$Q_\theta = M - kL^2 \sin \theta + mgL \cos \theta - 2mg \left( L \cos \theta + \frac{L}{2} \sin \frac{\theta}{2} \right) = M - kL^2 \sin \theta - mgL \cos \theta - mgL \sin \frac{\theta}{2}$$

$$\Rightarrow \boxed{\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = Q_\theta} \Rightarrow \left( 5 - 2 \sin \frac{\theta}{2} \right) \ddot{\theta} - \frac{1}{2} \cos \frac{\theta}{2} \dot{\theta}^2 = \frac{M}{ml^2} - \frac{k}{m} \sin \theta - \frac{g}{L} \cos \theta - \frac{g}{L} \sin \frac{\theta}{2}$$

3.2.a

$$\boxed{\frac{d}{dt} \bar{M}_D \Big|_{(\theta, \dot{\theta}, \ddot{\theta})}} = \sum \bar{m}_{e,D} + 2m \bar{v}_G \times \bar{v}_D \Big|_{(\underline{R_K}, mg)} \text{ pour la tige DE seule}$$

$$\text{avec } \begin{cases} \bar{M}_D = \bar{I}_D \cdot \ddot{\omega} + 2m \overline{DG} \times \bar{v}_D = \left( \frac{4}{3} ml^2 \ddot{\theta} - 2ml^2 \sin \frac{\theta}{2} \dot{\theta} \right) \bar{l}_z \\ \bar{m}_{e,D} = -2mgl \sin \frac{\theta}{2} \bar{l}_z + 2l \sin \frac{\theta}{2} R_K \bar{l}_z \end{cases}$$

$$ml^2 \ddot{\theta} \left( \frac{4}{3} - 2 \sin \frac{\theta}{2} \right) - ml^2 \dot{\theta}^2 \cos \frac{\theta}{2} = -2mgl \sin \frac{\theta}{2} + 2l \sin \frac{\theta}{2} R_K + ml^2 \dot{\theta}^2 \cos \frac{\theta}{2}$$

$$\Rightarrow R_K = \frac{ml}{2 \sin \frac{\theta}{2}} \left( \left( \frac{4}{3} - 2 \sin \frac{\theta}{2} \right) \ddot{\theta} - 2 \cos \frac{\theta}{2} \dot{\theta}^2 \right) + mg$$

3.2.b

$$\text{tige AB seule : } \boxed{\frac{d}{dt} \bar{M}_B \Big|_{(\theta, \dot{\theta}, \ddot{\theta})}} = \sum \bar{m}_{e,B} \Big|_{(\underline{X_H}, \underline{Y_H}, M, mg)}$$

$$\text{avec } \boxed{\bar{M}_B = \bar{I}_B \cdot \ddot{\omega} + 2m \overline{BG} \times \bar{v}_B} = \frac{2m(2L)^2}{3} \dot{\theta} \bar{l}_z - 2mL^2 \dot{\theta} (\cos^2 \theta + \sin^2 \theta) \bar{l}_z = \frac{2mL^2}{3} \dot{\theta} \bar{l}_z$$

$$\Rightarrow \frac{2mL^2}{3} \ddot{\theta} = M - 2mgL \cos \theta + Y_H L \cos \theta - X_H L \sin \theta \quad (1)$$

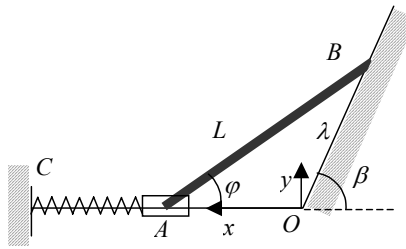
$$\text{tiges AB et BC : } \boxed{\frac{d}{dt} \bar{M}_C \Big|_{(\theta, \dot{\theta}, \ddot{\theta})}} = \sum \bar{m}_{e,C} \Big|_{(\underline{X_H}, \underline{Y_H}, M, mg)} + m \bar{v}_G \times \bar{v}_C \text{ avec } m_{AB} \underbrace{\bar{v}_{G_1} \times \bar{v}_C}_{=0} + m_{BC} \underbrace{\bar{v}_{G_2} \times \bar{v}_C}_{//}$$

$$\text{avec } \boxed{\bar{M}_C = \bar{M}_{C,AB} + \bar{M}_{C,BC}} \text{ et } \begin{cases} \bar{M}_{C,BC} = \cancel{\bar{I}_C} \cdot \ddot{\omega} + m \overline{CG_2} \times \bar{v}_C = m \frac{L}{2} \cdot L \dot{\theta} \cdot \cos \theta \bar{l}_z \\ \bar{M}_{C,AB} = \bar{M}_{G_1,AB} + \cancel{\overline{CG_1} \times \bar{R}_{AB}} = \frac{2m(2L)^2}{12} \dot{\theta} \bar{l}_z \end{cases}$$

$$\Rightarrow mL^2 \left( \frac{2}{3} + \frac{1}{2} \cos \theta \right) \ddot{\theta} - m \frac{L^2}{2} \sin \theta \dot{\theta}^2 = M + 2mg(L - L \cos \theta) + mg \frac{L}{2} - Y_H (L - L \cos \theta) - X_H L \sin \theta \quad (2)$$

2 équations, 2 inconnues

4.



$$T = \left( \frac{1}{2} m v_{G_i}^2 + \frac{1}{2} \bar{\omega} \cdot \bar{I}_{G_i} \cdot \bar{\omega} \right) = \frac{1}{2} m v_{G_{AB}}^2 + \frac{1}{2} \bar{\omega} \cdot \bar{I}_{G_{AB}} \cdot \bar{\omega}$$

Un degré de liberté  $\phi$ . On travaille avec une coordonnée généralisée donc la coordonnée  $\lambda$  doit être remplacée dans le Lagrangien avec la relation  $L \sin \phi = \lambda \sin \beta$ .

$$T = \frac{1}{2} m \left( L^2 \frac{\cos^2(\beta - \phi) \dot{\phi}^2}{\sin^2 \beta} + \frac{L^2}{4} \dot{\phi}^2 - L^2 \frac{\cos(\beta - \phi)}{\sin \beta} \sin \phi \dot{\phi}^2 \right) + \frac{1}{2} m \frac{L^2}{12} \dot{\phi}^2$$

4.1

$$\text{avec } \begin{cases} \overline{OG} = \left( \frac{L}{2} \cos \phi - \lambda \cos \beta \right) \bar{1}_x + \frac{L}{2} \sin \phi \bar{1}_y = \left( \frac{L}{2} \cos \phi - L \frac{\sin \phi \cos \beta}{\sin \beta} \right) \bar{1}_x + \frac{L}{2} \sin \phi \bar{1}_y \\ \overline{OG} = \left( \frac{L}{2} \cos \phi - L \cos \phi + L \frac{\sin(\beta - \phi)}{\sin \beta} \right) \bar{1}_x + \frac{L}{2} \sin \phi \bar{1}_y = \left( L \frac{\sin(\beta - \phi)}{\sin \beta} - \frac{L}{2} \cos \phi \right) \bar{1}_x + \frac{L}{2} \sin \phi \bar{1}_y \\ \bar{v}_G = \left( L \frac{\cos(\beta - \phi)(-\dot{\phi})}{\sin \beta} + \frac{L}{2} \sin \phi \dot{\phi} \right) \bar{1}_x + \left( \frac{L}{2} \cos \phi \dot{\phi} \right) \bar{1}_y \end{cases}$$

$$\Rightarrow T = \frac{1}{2} m \left( \frac{\cos^2(\beta - \phi)}{\sin^2 \beta} - \frac{\cos(\beta - \phi)}{\sin \beta} \sin \phi \right) L^2 \dot{\phi}^2 + \frac{1}{2} m \frac{L^2}{3} \dot{\phi}^2$$

$$Q_i = \sum_h \bar{F}_h \cdot \frac{\partial \bar{\varphi}_h}{\partial q_i}$$

$$\bullet \overline{OG} = \left( \frac{L}{2} \cos \phi - \lambda \cos \beta \right) \bar{1}_x + \frac{L}{2} \sin \phi \bar{1}_y \Rightarrow \delta \overline{OG} \Big|_y = \frac{L}{2} \cos \phi \delta \phi \text{ et } \bar{F}_G = -mg \bar{1}_y$$

$$\bullet \overline{OA} = L \left( \frac{\sin \beta \cos \phi - \sin \phi \cos \beta}{\sin \beta} \right) \bar{1}_x = L \left( \frac{\sin(\beta - \phi)}{\sin \beta} \right) \bar{1}_x$$

$$\Rightarrow \delta \overline{OA} = -L \left( \frac{\cos(\beta - \phi)}{\sin \beta} \right) \delta \phi \text{ et } \bar{F}_A = k \left( L - (L \cos \phi - \lambda \cos \beta) \right) \bar{1}_x$$

$$\bullet \overline{OB} = -\lambda \cos \beta \bar{1}_x + \lambda \sin \beta \bar{1}_y = -L \frac{\sin \phi}{\sin \beta} \cos \beta \bar{1}_x + L \sin \phi \bar{1}_y = \lambda \bar{1}_y = L \frac{\sin \phi}{\sin \beta} \bar{1}_y$$

$$\Rightarrow \delta \overline{OB} = L \frac{\cos \phi}{\sin \beta} \delta \phi \bar{1}_y \text{ et } \bar{F}_B = N_B \bar{1}_u - T_B \frac{\dot{\phi}}{|\dot{\phi}|} \bar{1}_v \text{ avec } \bar{T}_B = f N_B \frac{\dot{\phi}}{|\dot{\phi}|} \bar{1}_v$$

$$Q_\phi = \sum_i \bar{F}_i \cdot \frac{\partial \bar{\varphi}_i}{\partial q_\phi} \Rightarrow Q_\phi = -mg \frac{L}{2} \cos \phi - \frac{k L^2}{\sin^2 \beta} \cos(\beta - \phi) (\sin \beta - \sin(\beta - \phi)) - f N_B \frac{\dot{\phi}}{|\dot{\phi}|} L \frac{\cos \phi}{\sin \beta}$$

$$\left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}} \right) - \left( \frac{\partial T}{\partial \phi} \right) = Q_\phi \right]$$

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}} \right) - \left( \frac{\partial T}{\partial \phi} \right) &= \frac{d}{dt} \left( m \left( \frac{\cos^2(\beta - \phi)}{\sin^2 \beta} - \frac{\cos(\beta - \phi)}{\sin \beta} \sin \phi \right) L^2 \dot{\phi} + m \frac{L^2}{3} \dot{\phi} \right) \\ &\quad - \left( m \left( \frac{\cos(\beta - \phi) \sin(\beta - \phi)}{\sin^2 \beta} - \frac{1}{2} \frac{\sin(\beta - \phi)}{\sin \beta} \sin \phi - \frac{1}{2} \frac{\cos(\beta - \phi)}{\sin \beta} \cos \phi \right) L^2 \dot{\phi}^2 \right) \\ &= \left( m \left( \frac{\cos^2(\beta - \phi)}{\sin^2 \beta} - \frac{\cos(\beta - \phi)}{\sin \beta} \sin \phi \right) L^2 \ddot{\phi} + m \frac{L^2}{3} \ddot{\phi} \right. \\ &\quad \left. + m \left( \frac{2 \cos(\beta - \phi) \sin(\beta - \phi)}{\sin^2 \beta} - \frac{\sin(\beta - \phi)}{\sin \beta} \sin \phi - \frac{\cos(\beta - \phi)}{\sin \beta} \cos \phi \right) L^2 \dot{\phi}^2 \right. \\ &\quad \left. - \left( m \left( \frac{\cos(\beta - \phi) \sin(\beta - \phi)}{\sin^2 \beta} - \frac{1}{2} \frac{\sin(\beta - \phi)}{\sin \beta} \sin \phi - \frac{1}{2} \frac{\cos(\beta - \phi)}{\sin \beta} \cos \phi \right) L^2 \dot{\phi}^2 \right) \right) \end{aligned}$$

$$\begin{aligned}
&= m \left( \frac{1}{3} + \frac{\cos^2(\beta - \phi)}{\sin^2 \beta} - \frac{\cos(\beta - \phi)}{\sin \beta} \sin \phi \right) L^2 \ddot{\phi} + m \left( \frac{\cos(\beta - \phi) \sin(\beta - \phi)}{\sin^2 \beta} \right) L^2 \dot{\phi}^2 \\
&\quad - m \left( \frac{1}{2} \frac{\sin(\beta - \phi)}{\sin \beta} \sin \phi + \frac{1}{2} \frac{\cos(\beta - \phi)}{\sin \beta} \cos \phi \right) L^2 \dot{\phi}^2 \\
&\boxed{\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}} \right) - \left( \frac{\partial T}{\partial \phi} \right) = Q_\phi} \\
&m \left( \frac{1}{3} + \frac{\cos^2(\beta - \phi)}{\sin^2 \beta} - \frac{\cos(\beta - \phi)}{\sin \beta} \sin \phi \right) L^2 \ddot{\phi} + m \left( \frac{\cos(\beta - \phi) \sin(\beta - \phi)}{\sin^2 \beta} - \frac{\cos(\beta - 2\phi)}{2 \sin \beta} \right) L^2 \dot{\phi}^2 \\
&\quad = -mg \frac{L}{2} \cos \phi - \frac{kL^2}{\sin^2 \beta} \cos(\beta - \phi) (\sin \beta - \sin(\beta - \phi)) - f N_B \left| \frac{\dot{\phi}}{\phi} \right| L \frac{\cos \phi}{\sin \beta}
\end{aligned}$$

Théorème de la résultante cinétique :

$$\begin{aligned}
&\boxed{\frac{d}{dt} \bar{R} \Big|_x = \bar{F} \Big|_x} \\
&\frac{d}{dt} m \left( L \frac{\cos(\beta - \phi) (-\dot{\phi})}{\sin \beta} + \frac{L}{2} \sin \phi \dot{\phi} \right) = k \left( L - (L \cos \phi - \lambda \cos \beta) \right) + f N_B \cos \beta + N_B \sin \beta \\
&mL \left( \frac{1}{2} \sin \phi - \frac{\cos(\beta - \phi)}{\sin \beta} \right) \ddot{\phi} + mL \left( \frac{1}{2} \cos \phi - \frac{\sin(\beta - \phi)}{\sin \beta} \right) \dot{\phi}^2 \\
&\quad = k \left( L - (L \cos \phi - \lambda \cos \beta) \right) + (f \cos \beta + \sin \beta) N_B
\end{aligned}$$

$\Rightarrow N_B$  à remplacer dans l'équation de mouvement.

#### 4.2. 2 coordonnées généralisées $x_A$ et $\phi$

$$\begin{aligned}
&\boxed{T = \left( \frac{1}{2} m v_{G_i}^2 + \frac{1}{2} \bar{\omega} \cdot \bar{I}_{G_i} \cdot \bar{\omega} \right) = \frac{1}{2} m v_{G_{AB}}^2 + \frac{1}{2} \bar{\omega} \cdot \bar{I}_{G_{AB}} \cdot \bar{\omega}} \\
&\text{avec } \overline{OG} = \left( x_A - \frac{L}{2} \cos \phi \right) \bar{\mathbf{i}}_x + \frac{L}{2} \sin \phi \bar{\mathbf{i}}_y \Rightarrow \bar{\mathbf{v}}_G = \left( \dot{x}_A + \frac{L}{2} \sin \phi \dot{\phi} \right) \bar{\mathbf{i}}_x + \left( \frac{L}{2} \cos \phi \dot{\phi} \right) \bar{\mathbf{i}}_y \\
&T = \frac{1}{2} m \left( \dot{x}_A^2 + \frac{L^2}{4} \dot{\phi}^2 + L \dot{x}_A \sin \phi \dot{\phi} \right) + \frac{1}{2} m \frac{L^2}{12} \dot{\phi}^2 = \frac{1}{2} m \left( \dot{x}_A^2 + L \dot{x}_A \sin \phi \dot{\phi} \right) + \frac{1}{2} m \frac{L^2}{3} \dot{\phi}^2
\end{aligned}$$

1 contraintes liant les deux coordonnées généralisées

$$\boxed{\sum_{j=1}^p \lambda_j \frac{\partial \phi_j}{\partial q_i}}$$

$$\lambda_1 : x_A \sin \beta = L \sin(\beta - \phi) \Rightarrow x_A \sin \beta - L \sin(\beta - \phi) = 0 \Rightarrow \delta x_A \sin \beta + L \cos(\beta - \phi) \delta \phi = 0$$

$$\boxed{Q_i = \sum_h \bar{F}_h \cdot \frac{\partial \bar{\phi}_h}{\partial q_i}}$$

- $\overline{OG} = \left( x_A - \frac{L}{2} \cos \phi \right) \bar{\mathbf{i}}_x + \frac{L}{2} \sin \phi \bar{\mathbf{i}}_y \Rightarrow \delta \overline{OG} \Big|_y = \frac{L}{2} \cos \phi \delta \phi$  et  $\bar{F}_G = -mg \bar{\mathbf{i}}_y$
- $\overline{OA} = x_A \bar{\mathbf{i}}_x \Rightarrow \delta \overline{OA} = \delta x_A \bar{\mathbf{i}}_x$  et  $\bar{F}_A = k(L - x_A) \bar{\mathbf{i}}_x$
- $\overline{OB} = x_A \bar{\mathbf{i}}_x - L \cos \phi \bar{\mathbf{i}}_x + L \sin \phi \bar{\mathbf{i}}_y \Rightarrow \delta \overline{OB} = (\delta x_A + L \sin \phi \delta \phi) \bar{\mathbf{i}}_x + L \cos \phi \delta \phi \bar{\mathbf{i}}_y$

$$\bar{F}_B = N_B \bar{\mathbf{i}}_y - T_B \left| \frac{\dot{\phi}}{\phi} \right| \bar{\mathbf{i}}_y = \left( N_B \sin \beta + T_B \left| \frac{\dot{\phi}}{\phi} \right| \cos \beta \right) \bar{\mathbf{i}}_x + \left( N_B \cos \beta - T_B \left| \frac{\dot{\phi}}{\phi} \right| \sin \beta \right) \bar{\mathbf{i}}_y$$

$$\Rightarrow \begin{cases} Q_\phi = -mg \frac{L}{2} \cos \phi + L \sin \phi N_B \left( \sin \beta + f \frac{\dot{\phi}}{|\dot{\phi}|} \cos \beta \right) + L \cos \phi N_B \left( \cos \beta - f \frac{\dot{\phi}}{|\dot{\phi}|} \sin \beta \right) \\ Q_{x_A} = k(L - x_A) + N_B \left( \sin \beta + f \frac{\dot{\phi}}{|\dot{\phi}|} \cos \beta \right) \end{cases}$$

$$\Rightarrow \begin{cases} Q_\phi = -mg \frac{L}{2} \cos \phi + L N_B \left( \cos(\beta - \phi) - f \frac{\dot{\phi}}{|\dot{\phi}|} \sin(\beta - \phi) \right) \\ Q_{x_A} = k(L - x_A) + N_B \left( \sin \beta + f \frac{\dot{\phi}}{|\dot{\phi}|} \cos \beta \right) \end{cases}$$

$$\boxed{\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\phi}} \right) - \left( \frac{\partial T}{\partial \phi} \right) = Q_\phi + \sum_i \lambda_i \frac{\partial \phi_i}{\partial q_j}}$$

$$\left\{ \begin{aligned} m \left( \frac{L}{2} \ddot{x}_A \sin \phi + \frac{L}{2} \dot{x}_A \cos \phi \dot{\phi} + \frac{L^2}{3} \ddot{\phi} - \frac{L}{2} \dot{x}_A \cos \phi \dot{\phi} \right) &= m \left( \frac{L^2}{3} \ddot{\phi} + \frac{L}{2} \ddot{x}_A \sin \phi \right) = \\ &-mg \frac{L}{2} \cos \phi + L N_B \left( \cos(\beta - \phi) - f \frac{\dot{\phi}}{|\dot{\phi}|} \sin(\beta - \phi) \right) + \lambda_1 L \cos(\beta - \phi) \quad (1) \\ m \ddot{x}_A + \frac{1}{2} m L \sin \phi \ddot{\phi} + \frac{1}{2} m L \cos \phi \dot{\phi}^2 &= k(L - x_A) + N_B \left( \sin \beta + f \frac{\dot{\phi}}{|\dot{\phi}|} \cos \beta \right) + \lambda_1 \sin \beta \quad (2) \\ x_A \sin \beta = L \sin(\beta - \phi); \quad \dot{x}_A \sin \beta &= -L \cos(\beta - \phi) \dot{\phi}; \quad \ddot{x}_A \sin \beta = -L \sin(\beta - \phi) \dot{\phi}^2 - L \cos(\beta - \phi) \ddot{\phi} \quad (3) \end{aligned} \right. \quad \text{On}$$

remplace la coordonnée  $x_A$  ainsi que ses dérivées dans les équations (1) et (2) par les équations (3)

On isole le multiplicateur de Lagrange de l'équation (2) et on le réinjecte dans l'équation (1).

On retrouve la même équation de mouvement qu'au (4).1 après quelques remplacements.

$$mL^2 \left( \frac{1}{3} + \frac{\cos^2(\beta - \phi)}{\sin^2 \beta} - \sin \phi \frac{\cos(\beta - \phi)}{\sin \beta} \right) \ddot{\phi} + mL^2 \left( \frac{\cos(\beta - \phi) \sin(\beta - \phi)}{\sin^2 \beta} - \frac{\cos(\beta - 2\phi)}{2 \sin \beta} \right) \dot{\phi}^2$$

$$= -mg \frac{L}{2} \cos \phi - kL \frac{L \cos(\beta - \phi)}{\sin \beta} (\sin \beta - \sin(\beta - \phi)) - N_B L f \frac{\dot{\phi}}{|\dot{\phi}|} \frac{\cos \phi}{\sin \beta}$$

**5.1** 1 degré de liberté , 3 coordonnées  $\{z, \theta_1, \theta_2\}$

**5.2**  $T = \left[ \frac{1}{2} M \dot{z}^2 \right] + \left[ \frac{1}{2} m L^2 (\sin^2 \theta_1 \omega^2 + \dot{\theta}_1^2) \right] + \left[ \frac{1}{2} m L^2 (\sin^2 \theta_2 \omega^2 + \dot{\theta}_2^2) \right]$  et  $V = -Mgz - mgL(\cos \theta_1 + \cos \theta_2)$

avec  $\begin{cases} \bar{v}_{m1} = \bar{\omega}_1 \times \overline{OM_1} \text{ et } \bar{\omega}_1 = \omega \bar{1}_z - \dot{\theta}_1 \bar{1}_x \Rightarrow \bar{v}_{m1} = -L \cos \theta_1 \dot{\theta}_1 \bar{1}_y + L \sin \theta_1 \dot{\theta}_1 \bar{1}_z + L \sin \theta_1 \omega \bar{1}_x \\ \bar{v}_{m2} = \bar{\omega}_2 \times \overline{OM_2} \text{ et } \bar{\omega}_2 = \omega \bar{1}_z + \dot{\theta}_2 \bar{1}_x \Rightarrow \bar{v}_{m2} = +L \cos \theta_2 \dot{\theta}_2 \bar{1}_y + L \sin \theta_2 \dot{\theta}_2 \bar{1}_z - L \sin \theta_2 \omega \bar{1}_x \end{cases}$

$$\Rightarrow L = \frac{1}{2} M \dot{z}^2 + \frac{1}{2} m L^2 (\sin^2 \theta_1 + \sin^2 \theta_2) \omega^2 + \frac{1}{2} m L^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) + Mgz + mgL(\cos \theta_1 + \cos \theta_2)$$

Deux contraintes :  $\begin{cases} b^2 = z^2 + a_1^2 - 2a_1 z \cos \theta_1 \Rightarrow \lambda_1 (z \delta z - a_1 \delta z \cos \theta_1 + a_1 z \sin \theta_1 \delta \theta_1 = 0) \\ b^2 = z^2 + a_2^2 - 2a_2 z \cos \theta_2 \Rightarrow \lambda_2 (z \delta z - a_2 \delta z \cos \theta_2 + a_2 z \sin \theta_2 \delta \theta_2 = 0) \end{cases}$

$$\Rightarrow \left\{ \begin{array}{l} \frac{d}{dt} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = \sum_{j=1}^p \lambda_j \frac{\partial \phi_j}{\partial x} : M\ddot{z} - Mg = \lambda_1 (z - a_1 \cos \theta_1) + \lambda_2 (z - a_2 \cos \theta_2) \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = \sum_{j=1}^p \lambda_j \frac{\partial \phi_j}{\partial \theta_1} : mL^2 \ddot{\theta}_1 - mL^2 \omega^2 \sin \theta_1 \cos \theta_1 + mgL \sin \theta_1 = \lambda_1 a_1 z \sin \theta_1 \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = \sum_{j=1}^p \lambda_j \frac{\partial \phi_j}{\partial \theta_2} : mL^2 \ddot{\theta}_2 - mL^2 \omega^2 \sin \theta_2 \cos \theta_2 + mgL \sin \theta_2 = \lambda_2 a_2 z \sin \theta_2 \\ z\ddot{z} - a_1 \cos \theta_1 \dot{z} + a_1 z \sin \theta_1 \dot{\theta}_1 = 0 \\ z\ddot{z} - a_2 \cos \theta_2 \dot{z} + a_2 z \sin \theta_2 \dot{\theta}_2 = 0 \end{array} \right.$$

**5.3** Dans ce cas, nous n'avons plus qu'un degré de liberté : 2 équations de Lagrange avec 1 multiplicateur.

$$\left\{ \begin{array}{l} a_1 = a_2 = b \Rightarrow \theta_1 = \theta_2 \text{ et } \lambda_1 = \lambda_2 \\ z\ddot{z} - b \cos \theta \dot{z} + bz \sin \theta \dot{\theta} = 0 \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = \sum_{j=1}^p \lambda_j \frac{\partial \phi_j}{\partial x} : M\ddot{z} - Mg = 2\lambda_1 (z - b \cos \theta) \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \sum_{j=1}^p \lambda_j \frac{\partial \phi_j}{\partial \theta} : mL^2 \ddot{\theta} - mL^2 \omega^2 \sin \theta \cos \theta + mgL \sin \theta = \lambda_1 bz \sin \theta \end{array} \right.$$

$$z = 2b \cos \theta ; \dot{z} = -2b \sin \theta \dot{\theta} ; \ddot{z} = -2b \cos \theta \dot{\theta}^2 - 2b \sin \theta \ddot{\theta}$$

$$\Rightarrow mL^2 \ddot{\theta} - mL^2 \omega^2 \sin \theta \cos \theta + mgL \sin \theta = \frac{M\ddot{z} - Mg}{2(z - b \cos \theta)} bz \sin \theta$$

$$= \frac{M(-2b \cos \theta \dot{\theta}^2 - 2b \sin \theta \ddot{\theta}) - Mg}{2(2b \cos \theta - b \cos \theta)} b 2b \cos \theta \sin \theta = (M(-2b \cos \theta \dot{\theta}^2 - 2b \sin \theta \ddot{\theta}) - Mg) b \sin \theta$$

$$\Rightarrow (mL^2 + 2Mb^2 \sin^2 \theta) \ddot{\theta} + (2Mb^2 \cos \theta \sin \theta) \dot{\theta}^2 - mL^2 \omega^2 \sin \theta \cos \theta + (mL + Mb) g \sin \theta = 0$$

Pour les problèmes relatifs au Tps et aux laboratoires, contactez [Emmanuelle.Vin@ulb.ac.be](mailto:Emmanuelle.Vin@ulb.ac.be)

Pour les problèmes relatifs aux projets Matlab, contactez [CFAO.Matlab@ulb.ac.be](mailto:CFAO.Matlab@ulb.ac.be)

Les énoncés et les corrigés sont accessibles et mis à jour sont sur le site de méca :

<http://beams.ulb.ac.be/beams/teaching/meca200/tps.html>