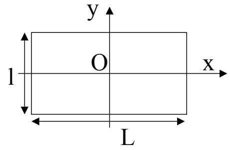
1.1

$$I_x = \rho \int_{-L/2}^{L/2} dx \int_{-l/2}^{l/2} y^2 dy = \rho \frac{Ll^3}{12} = m\frac{l^2}{12} \Rightarrow I_y = m\frac{L^2}{12}$$



1.2

$$I_y^{(a)} = \rho\frac{(4a)(4a)^3}{12} - \left[\rho\frac{(3a)(2a)^3}{12}\right] = \frac{58}{3}\rho a^4 = I_y^{(b)} = I_y^{(c)} \Rightarrow r_y^{(a)} = r_y^{(b)} = r_y^{(c)} = \sqrt{\frac{29}{15}}\,a$$

$$I_x^{(a)} = \rho\frac{(4a)(4a)^3}{12} - \left[\rho\frac{(2a)(3a)^3}{12} + \rho(3a)(2a)\left(\frac{a}{2}\right)^2\right] = \frac{46}{3}\rho a^4 \Rightarrow r_x^{(a)} = \sqrt{\frac{23}{15}}\,a$$

$$I_x^{(b)} = \rho\frac{(4a)(4a)^3}{12} - 2.\left[\rho\frac{(2a)(\frac{3a}{2})^3}{12} + \rho(\frac{3a}{2})(2a)\left(\frac{5a}{4}\right)^2\right] = \frac{65}{6}\rho a^4 \Rightarrow r_x^{(b)} = \sqrt{\frac{13}{12}}\,a$$

$$I_x^{(c)} = \rho\frac{(4a)(4a)^3}{12} - \left[\rho\frac{(2a)(3a)^3}{12}\right] = \frac{101}{6}\rho a^4 \Rightarrow r_x^{(c)} = \sqrt{\frac{101}{60}}\,a$$

2.

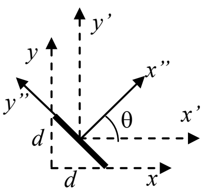
- $I_x^{(a)} = \int_{-\frac{a}{2}}^{+\frac{a}{2}} \rho y^2 dx \underbrace{=}_{y=0} 0;\ I_y^{(a)} = \int_{-\frac{a}{2}}^{+\frac{a}{2}} \rho x^2 dx = \rho\frac{a^3}{12} \Rightarrow \boxed{I_y^{(a)} = \frac{ma^2}{12}}$ et $P_{xy}^{(a)} = \int_{-\frac{a}{2}}^{+\frac{a}{2}} \rho xy dx = 0$ car $y = 0$

- $$\begin{cases} I_x^{(b)} = \int_{-\frac{a}{2}}^{+\frac{a}{2}} \rho y^2 dx \underbrace{=}_{y=a} \rho a^3 = \underbrace{I_{x_G}^{(b)}}_{=0} + ma^2 = ma^2;\ I_y^{(b)} = \int_{-\frac{a}{2}}^{+\frac{a}{2}} \rho x^2 dx = \frac{ma^2}{12} = I_{y_G}^{(b)} \text{ et} \\ P_{xy}^{(b)} = \int_{-\frac{a}{2}}^{+\frac{a}{2}} \rho xy dx \underbrace{=}_{y=a} 0 = P_{x_G y_G} + m.0.a \end{cases}$$

- $I_x^{(c)} = 0 + ma^2;\ I_y^{(c)} = \frac{ma^2}{12} + m(a)^2 = \frac{13ma^2}{12};\ P_{xy}^{(c)} = 0 + m.(a).(-a) = -ma^2$

- $I_x^{(d)} = 0 + m(2a)^2;\ I_y^{(d)} = \frac{ma^2}{12} + m\left(\frac{3a}{2}\right)^2 = \frac{7ma^2}{3};\ P_{xy}^{(d)} = 0 + m.\left(\frac{3a}{2}\right).(2a) = 3ma^2$

- $$\begin{cases} I_x^{(e)} = I_y^{(e)} \text{ Par symétrie} \qquad I_x^{(e)} = \int y^2 dm = I_{x'}^{(e)} + m\left(\frac{a}{2}.\sin 45\right)^2 \\ I_{x'}^{(e)} = \int y'^2\, dm = \int (\sin\theta x'' + \cos\theta y'')^2\, dm = \\ I_{x'}^{(e)} = \rho\int (\cos^2\theta y''^2 + \sin^2\theta x''^2 + 2\cos\theta\sin\theta x'' y'')dy' \underbrace{=}_{x''=0} \rho\cos^2\theta \int_{-\frac{a}{2}}^{\frac{a}{2}} y''^2\, dy'' \\ I_{x'}^{(e)} = \cos^2\theta.I_{x''}^{(e)} = \cos^2\theta\frac{ma^2}{12} \underbrace{=}_{\cos^2\theta = \frac{1}{2}} \frac{ma^2}{24} \Rightarrow I_x^{(e)} = \frac{ma^2}{24} + m\left(\frac{a}{2}\frac{\sqrt{2}}{2}\right)^2 = \frac{ma^2}{6} \\ P_{x'}^{(e)} = \int (\cos\theta x'' - \sin\theta y'')(\sin\theta x'' + \cos\theta y'')dm \underbrace{=}_{x'=0} -\sin\theta\cos\theta\int_{-\frac{a}{2}}^{\frac{a}{2}} y''^2\, dm \\ \quad = -\sin\theta\cos\theta.I_{x''}^{(e)} = -\frac{1}{2}\frac{ma^2}{12} \Rightarrow P_x^{(e)} = P_{x'}^{(e)} + m\left(\frac{a}{2}\frac{\sqrt{2}}{2}\right)\left(\frac{a}{2}\frac{\sqrt{2}}{2}\right) = \frac{ma^2}{12} \end{cases}$$



(e)

3.1

cercle : $I_z(\text{cercle}) = \int (x'^2 + y'^2)dm = \int_0^{2\pi} R^2 \rho R d\theta = M_{Cercle}R^2 = I_x(\text{cercle}) + I_y(\text{cercle})$ en 2D

$I_z(\text{cercle}) = 2I_x(\text{cercle})$ (par symétrie) $\Rightarrow I_x(\text{cercle}) = \frac{M_{Cercle}R^2}{2}$

Par symétrie : $I_x(\text{cercle O}) = I_x(\text{demi-cercle} \cup) + I_x(\text{demi-cercle} \cap) \Rightarrow I_x(\cup) = \frac{M_O R^2}{4} = \frac{M_\cup R^2}{2}$

3.2 Disque : $I_z\,(\text{Disque O}) = \int \left(x'^2 + y'^2\right) dm = \int_0^R \int_0^{2\pi} r^2 \rho r dr d\theta = \rho \frac{R^4}{4} 2\pi = \frac{M_{Disque} R^2}{2}$

$$\boxed{I_z(\text{O}) \underbrace{=}_{\text{en 2D}} I_x(\text{O}) + I_y(\text{O}) \underbrace{=}_{\textit{symétrie}} 2I_x(\text{O})} \Rightarrow I_x(\text{O}) = \frac{M_{\text{O}} R^2}{4}$$

Par symétrie : $I_x(\text{O}) = I_x(\cup) + I_x(\cap) \Rightarrow I_x(\cup) = \frac{M_{\text{O}} R^2}{8} = \frac{M_{\cup} R^2}{4}$

3.3 Sphère : $I_z\,(\text{Sphère O}) = \int \left(x'^2 + y'^2\right) dm = \int_0^R \left(\int_0^{\pi} \left(\int_0^{2\pi} \left(r \sin\theta\right)^2 \rho r^2 \sin\theta d\varphi \right) d\theta \right) dr = \frac{2}{5} M_{Sphère} R^2$

$$\boxed{I_z\,(\text{Sphère O}) = I_x(\text{O}) + I_y(\text{O}) - 2I_{xy}(\text{O})}$$

avec $\boxed{I_{xy}(\text{O}) = \int z^2 dm} = \int_0^R \left(\int_0^{\pi} \left(\int_0^{2\pi} \left(r \cos\theta\right)^2 \rho r^2 \sin\theta d\varphi \right) d\theta \right) dr = \frac{1}{5} M_{Sphère} R^2$

$I_x(\text{O}) = I_y(\text{O})$ (par symétrie) $\Rightarrow I_x(\text{O}) = \frac{I_z(\text{O})}{2} + I_{xy}(\text{O}) = \frac{1}{5} M_{Sphère} R^2 + \frac{1}{5} M_{Sphère} R^2 = \frac{2}{5} M_{Sphère} R^2$

On pouvait évidemment trouver directement $I_x = I_y = I_z$ par symétrie.

Par symétrie : $I_x(\text{O}) = I_x(\cup) + I_x(\cap) \Rightarrow I_x(\cup) = \frac{M_{Sphère} R^2}{5} = \frac{2 M_{\cup} R^2}{5}$

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4. Soient 1=*AB*, 2=*DE*, 3=*BCD*

$$I_{x1} = \frac{m_1 l_1^2}{3} . \frac{1}{2} = \frac{9}{2} \rho\, cm^3 \quad et \quad I_{y1} = \frac{m_1 l_1^2}{12} . \frac{1}{2} + m_1 \left(\frac{l_1}{2} \frac{\sqrt{2}}{2} + 2 \right)^2 = \rho \left(\frac{33}{2} + 9\sqrt{2} \right) cm^3;$$

$$I_{x2} = \frac{m_2 l_2^2}{3} . \frac{3}{4} = \frac{9}{2} \frac{\sqrt{2}}{\sqrt{3}} \rho\, cm^3 \quad et \quad I_{y2} = \frac{m_2 l_2^2}{12} . \frac{1}{4} + m_2 \left(\frac{l_1}{2} \frac{1}{2} + 2 \right)^2 = \rho \left(\frac{27}{2} \frac{\sqrt{2}}{\sqrt{3}} + 6 \right) cm^3;$$

$$I_{x3} = \frac{m_3 R^2}{2} - m_3 \left(\frac{2R}{\pi} \right)^2 + m_3 \left(\frac{2R}{\pi} + 3 \frac{\sqrt{2}}{2} \right)^2 = \rho\,(13\pi + 24\sqrt{2}) cm^3 \quad et \quad I_{y3} = \frac{m_3 R^2}{2} = 4\pi\rho\;\; cm^3;$$

$$I_x = \rho \left[\frac{9}{2} \left(1 + \frac{\sqrt{2}}{\sqrt{3}} \right) + 13\pi + 24\sqrt{2} \right] cm^3 = 82{,}956 \rho\;\; cm^3$$

$$I_x = \rho \left[\frac{45}{2} + 9\sqrt{2} + \frac{27}{2} \frac{\sqrt{2}}{\sqrt{3}} + 4\pi \right] cm^3 = 58{,}817 \rho\;\; cm^3$$

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5. Dans les axes liés au disque, nous avons

$$\overline{\overline{I}}_{GXYZ} = \begin{pmatrix} \frac{mR^2}{4} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{pmatrix}$$

La matrice de changement de base vaut :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1^1 & \alpha_2^1 & \alpha_3^1 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$\overline{1}_x = (1;0;0) = \left(\alpha_1^1;\alpha_2^1;\alpha_3^1\right)$$

$$\overline{1}_y = (0;\cos\alpha;\sin\alpha) = \left(\alpha_1^2;\alpha_2^2;\alpha_3^2\right)$$

$$\overline{1}_z = (0;-\sin\alpha;\cos\alpha) = \left(\alpha_1^3;\alpha_2^3;\alpha_3^3\right)$$

Donc :

$$I'_x = \alpha_i^1\alpha_j^1 I^{ij} = \left(\alpha_1^1\right)^2 I_X + \left(\alpha_2^1\right)^2 I_Y + \left(\alpha_3^1\right)^2 I_Z - 2\alpha_1^1\alpha_2^1 P_{XY} - 2\alpha_1^1\alpha_3^1 P_{XZ} - 2\alpha_2^1\alpha_3^1 P_{YZ}$$

$$I'_x = I_X = \frac{mR^2}{4}$$

$$I'_y = \alpha_i^2\alpha_j^2 I^{ij} = \cos^2\alpha\frac{mR^2}{4} + \sin^2\alpha\frac{mR^2}{2} = \frac{mR^2}{4}\left(1+\sin^2\alpha\right)$$

$$I'_z = \alpha_i^3\alpha_j^3 I^{ij} = \sin^2\alpha\frac{mR^2}{4} + \cos^2\alpha\frac{mR^2}{2} = \frac{mR^2}{4}\left(1+\cos^2\alpha\right)$$

$$-P'_{xy} = \alpha_i^1\alpha_j^2 I^{ij} = \left(\alpha_1^1\alpha_1^2 I_X + \alpha_2^1\alpha_2^2 I_Y + \alpha_3^1\alpha_3^2 I_Z - 2\alpha_1^1\alpha_2^2 P_{XY} - 2\alpha_1^1\alpha_3^2 P_{XZ} - 2\alpha_2^1\alpha_3^2 P_{YZ}\right) = 0$$

et $-P'_{xz} = \alpha_i^1\alpha_j^3 I^{ij} = 0$

Les axes Ox et OX sont confondu et forme une direction principale. => les produits d'inertie comprenant l'axe x sont nuls.

$$-P'_{yz} = I'^{23} = \alpha_i^2\alpha_j^3 I^{ij} = \alpha_2^2\alpha_2^3 I_Y + \alpha_3^2\alpha_3^3 I_Z = -\sin\alpha\cos\alpha\frac{mR^2}{4} + \sin\alpha\cos\alpha\frac{mR^2}{2} = m\frac{R^2}{4}\cos\alpha\sin\alpha$$

$$\overline{\overline{I}}_{Gxyz} = \begin{pmatrix} \frac{mR^2}{4} & 0 & 0 \\ 0 & \frac{mR^2}{4}\left(1+\sin^2\alpha\right) & m\frac{R^2}{4}\cos\alpha\sin\alpha \\ 0 & m\frac{R^2}{4}\cos\alpha\sin\alpha & \frac{mR^2}{4}\left(1+\cos^2\alpha\right) \end{pmatrix}$$