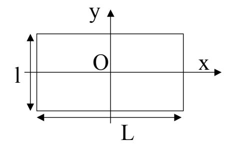
1.1

$$I_x = \rho \int_{-L/2}^{L/2} dx \int_{-l/2}^{l/2} y^2 dy = \rho \frac{Ll^3}{12} = m\frac{l^2}{12} \Rightarrow I_y = m\frac{L^2}{12}$$



1.2

$$I_x^{(a)} = \rho\frac{3a.(2a)^3}{12} - \left[\rho\frac{a.(a)^3}{12}\right] = \frac{23}{12}\rho a^4 = mr_x^2$$

$$\Rightarrow r_x{}^{(b)} = \sqrt{\frac{23}{60}}a \quad (m = \rho 5a^2)$$

$$I_y^{(a)} = \rho\frac{2a.(3a)^3}{12} - \left[\rho\frac{a.(a)^3}{12}\right] = \frac{53}{12}\rho a^4 = mr_y^2 \Rightarrow r_y{}^{(b)} = \sqrt{\frac{53}{60}}a$$

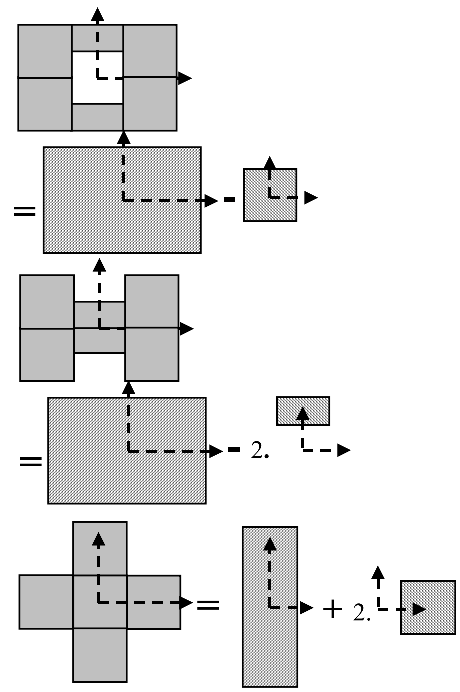
$$I_x^{(b)} = \rho\frac{3a.(2a)^3}{12} - 2.\left[\rho\frac{a.\left(\frac{a}{2}\right)^3}{12} + \rho a.\frac{a}{2}.\left(\frac{3a}{4}\right)^2\right] = \frac{17}{12}\rho a^4$$

$$\Rightarrow r_x{}^{(b)} = \sqrt{\frac{17}{60}}a$$

$$I_y^{(b)} = \rho\frac{2a.(3a)^3}{12} - 2.\left[\rho\frac{\frac{a}{2}.(a)^3}{12}\right] = \frac{53}{12}\rho a^4 \Rightarrow r_y{}^{(b)} = \sqrt{\frac{53}{60}}a$$

$$I_x^{(c)} = \rho\frac{a.(3a)^3}{12} + 2.\left[\rho\frac{a.(a)^3}{12}\right] = \frac{29}{12}\rho a^4 \Rightarrow r_x^{(c)} = \sqrt{\frac{29}{60}}\,a$$

$$I_y^{(c)} = I_x^{(c)} \Rightarrow r_y^{(c)} = r_x^{(c)}$$



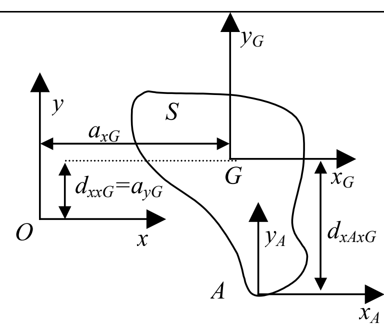
Rappel : Utilisation de la formule de Steiner

Pour calculer le moment d'inertie du solide S par rapport à l'axe x passant par O :

$$I_x = I_{x_G} + md_{xx_G}^2$$

Parfois, il est plus facile de calculer le moment d'inertie par rapport à un axe ne passant pas par G :

$$I_{x_A} = I_{x_G} + md_{x_A x_G}^2 \Rightarrow I_x = I_{x_G} + md_{xx_G}^2 = I_{x_A} + m\left(d_{xx_G}^2 - d_{x_A x_G}^2\right)$$



2. Rappel :

$$P_{xy} = \int xydm = \int\left(x_G + a_{x_G}\right)\left(y_G + a_{y_G}\right)dm = P_{x_G y_G} + ma_{xx_G}a_{y_G} + a_{x_G}\underbrace{\int y_G dm}_{=0} + a_{y_G}\underbrace{\int x_G dm}_{=0}$$

$$\text{ou } I_O^{\alpha\beta} = I_G^{\alpha\beta} + M\left(a^2\delta^{\alpha\beta} - a^\alpha a^\beta\right) \Rightarrow -P_{xy} = -P_{x_G y_G} - Ma^x a^y \Rightarrow P_{xy} = P_{x_G y_G} + ma_{x_G}a_{y_G} \quad \text{avec } x = x_G + a^x$$

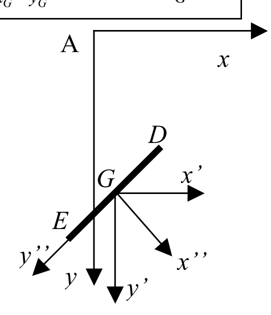
2.1

$$I_x(Tige_{AB}) = \int_0^a \rho y^2 dy = \rho\frac{a^3}{3} = \frac{ma^2}{3};\ I_y(Tige_{AB}) = 0 \text{ et } P_{xy}(Tige_{AB}) = 0$$

$$I_x(Tige_{BC}) = \rho a^3;\ I_y(Tige_{BC}) = \int_0^a \rho x^2 dx = \rho\frac{a^3}{3} = \frac{ma^2}{3} \text{ et } P_{xy}(Tige_{BC}) = \rho a.a.\frac{a}{2}$$

$$I_x(Tige_{CD}) = \int_a^{2a} \rho y^2 dy = \rho\left(\frac{8a^3}{3} - \frac{a^3}{3}\right) = \frac{7}{3}ma^2;\ I_y(Tige_{CD}) = \rho a^3$$

$$\text{et } P_{xy}(Tige_{CD}) = \rho a.\frac{3a}{2}.a$$



$$\begin{cases} I_{x''}(Tige_{DE}) = \int_{-a}^{+a} \rho y''^2\, dy = \rho \dfrac{2.a^3}{3} \\ I_{x'}(Tige_{DE}) = \int y'^2\, dm = \int \left(y''\dfrac{\sqrt{2}}{2} + x''\dfrac{\sqrt{2}}{2} \right)^2 dm = \dfrac{I_{x''}(Tige_{DE})}{2} + \underbrace{\dfrac{I_{y''}(Tige_{DE})}{2}}_{=0} + \underbrace{P_{x''y''}(Tige_{DE})}_{=0} \\ => I_x(Tige_{DE}) = I_{x'}(Tige_{DE}) + \rho 2a \left(2a + a\dfrac{\sqrt{2}}{2} \right)^2 = \rho \left(\dfrac{28}{3} a^3 + 4\sqrt{2} a^3 \right) \end{cases}$$

$$\begin{cases} I_{y'}(Tige_{DE}) = \int x'^2\, dm = \int \left(x''\dfrac{\sqrt{2}}{2} - y''\dfrac{\sqrt{2}}{2} \right)^2 dm = \dfrac{I_{x''}(Tige_{DE})}{2} + \underbrace{\dfrac{I_{y''}(Tige_{DE})}{2}}_{=0} - \underbrace{P_{x''y''}(Tige_{DE})}_{=0} = \rho \dfrac{a^3}{3} \\ I_y(Tige_{DE}) = I_{y'}(Tige_{DE}) + \rho 2a \left(a - a\dfrac{\sqrt{2}}{2} \right)^2 = \rho \dfrac{5.a^3}{3} \end{cases}$$

$$\begin{cases} P_{x'y'}(Tige_{DE}) = \int \left(x''\dfrac{\sqrt{2}}{2} - y''\dfrac{\sqrt{2}}{2} \right)\left(y''\dfrac{\sqrt{2}}{2} + x''\dfrac{\sqrt{2}}{2} \right) dm = \dfrac{P_{x''y''}}{2} + \underbrace{\dfrac{I_{y''}}{2}}_{=0} - \dfrac{I_{x''}}{2} - \dfrac{P_{x''y''}}{2} = -\dfrac{I_{x''}}{2} = -\rho \dfrac{a^3}{3} \\ P_{xy}(Tige_{DE}) = P_{x'y'}(Tige_{DE}) + \rho 2a. \left(2a + a\dfrac{\sqrt{2}}{2} \right) . \left(a - a\dfrac{\sqrt{2}}{2} \right) \end{cases}$$

2.2 $I_{x'}(Arc_{DE}) = \int y'^2\, dm = \int_0^{\pi} R^2 \sin^2\theta \rho a d\theta = \rho a^3 \dfrac{\pi}{2} = \dfrac{Ma^2}{2}$

$$I_{x_G}(Arc_{DE}) = \frac{Ma^2}{2} - M\left(\frac{2a}{\pi}\right)^2 \quad => \quad I_x(Arc_{DE}) = \frac{Ma^2}{2} - M\left(\frac{2a}{\pi}\right)^2 + M\left(2a + \frac{2a}{\pi}\right)^2$$

$$I_y(Arc_{DE}) = I_{y'}(Arc_{DE}) = \int x'^2\, dm = \int_0^{\pi} R^2 \cos^2\theta \rho R d\theta = \frac{Ma^2}{2}$$

$P_{x'y'}(Arc_{DE}) = 0$ par symétrie autour de y

cercle : $I_z(\text{cercle}) = \int \left(x'^2 + y'^2 \right) dm = \int_0^{\pi} R^2 \rho R d\theta = M_{Cercle} R^2 = I_x(\text{cercle}) + I_y(\text{cercle})$ en 2D
$I_z(\text{cercle}) = 2 I_x(\text{cercle})$ (par symétrie) => $I_x(\text{cercle}) = \dfrac{M_{Cercle} R^2}{2}$
Par symétrie : $I_x(\text{cercle O}) = I_x(\text{demi-cercle } \cup) + I_x(\text{demi-cercle } \cap)$ => $I_x(\cup) = \dfrac{M_O R^2}{4} = \dfrac{M_\cup R^2}{2}$

3.1	$I_y = \rho \int_0^b x^2 dx \int_0^{h\left(1-\frac{x}{b}\right)} dy = \rho \dfrac{hb^3}{12} = \dfrac{Mb^2}{6} => I_x = \dfrac{Mh^2}{6}$
3.2	$P_{xy} = \rho \int_0^b x dx \dfrac{h^2}{2}\left(1 - \dfrac{x}{b}\right)^2 = \rho \dfrac{h^2 b^2}{24} = \dfrac{Mbh}{12}$
3.3	si $b = 4\ cm$ et $h = 6\ cm$: $I_x = 6M\ cm^2$; $I_y = \dfrac{8}{3} M\ cm^2 \approx 2.67 M\ cm^2$; $P_{xy} = 2M\ cm^2$ A.P.I. en O : x', y' $\Big\vert$ $tg 2\theta = \dfrac{2P_{xy}}{I_y - I_x} = -1.2$ => $\theta \approx -25.1°$
3.4	Moment principaux d'inertie en O : $I_{x'} \approx 6.94 M\ cm^2$; $I_{y'} \approx 1.73 M\ cm^2$ $I_{y'} < I_y < I_x < I_{x'}$

3.5 Vecteur unitaire nécessaire pour faire la rotation du système d'axes $Ax'y'z'(//Oxyz)$ vers $Ax"y"z"$:

$$\overline{1}_{AC} = \frac{1}{\sqrt{3}}(-1;+1;+1) = (\alpha_1^1;\alpha_2^1;\alpha_3^1)$$

$$I_{AC} = I_{x"} = \alpha_i^1\alpha_j^1 I^{ij} = (\alpha_1^1)^2 I_{x'} + (\alpha_2^1)^2 I_{y'} + (\alpha_3^1)^2 I_{z'} - 2\alpha_1^1\alpha_2^1 P_{x'y'} - 2\alpha_1^1\alpha_3^1 P_{x'z'} - 2\alpha_2^1\alpha_3^1 P_{y'z'}$$

$$\Rightarrow\ I_{AC} = \frac{1}{3}I_{x'} + \frac{1}{3}I_{y'} + \frac{1}{3}I_{z'} + \frac{2}{3}P_{x'y'} + \frac{2}{3}\underbrace{P_{x'z'}}_{0} - \frac{2}{3}\underbrace{P_{y'z'}}_{=0}$$

Calcul des composantes du tenseur dans le repère $Ax'y'z'(//Oxyz)$

$$I_{x'} = I_x = \frac{Mb^2}{6};\ I_{y'} = I_y - md_{yy_G} + md_{y_Gy'} = \left[\frac{Mb^2}{6} - M\left(\frac{b}{3}\right)^2\right] + M\left(\frac{2b}{3}\right)^2 = \frac{Mb^2}{2};$$

$$I_{z'} = I_{x'} + I_{y'} = \frac{2Mb^2}{3}\ \ (\text{car } z = 0 \text{ pour la plaque});\ P_{x'y'} = \left[\frac{Mb^2}{12} - M\left(-\frac{b}{3}\right)\left(-\frac{b}{3}\right)\right] + M\left(\frac{2b}{3}\right)\left(-\frac{b}{3}\right) = -\frac{Mb^2}{4}$$

$$\Rightarrow I_{AC} = \frac{1}{3}\frac{Mb^2}{6} + \frac{1}{3}\frac{Mb^2}{2} + \frac{1}{3}\frac{2Mb^2}{3} - \frac{2}{3}\frac{Mb^2}{4} = \frac{5}{18}Mb^2$$

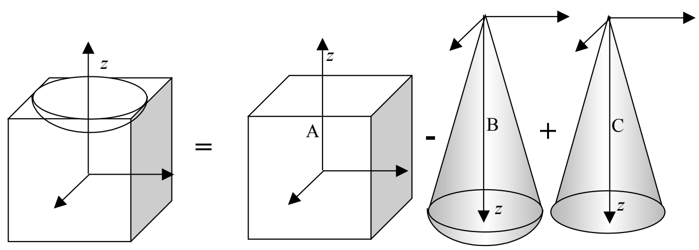
4.1

$$I_z(A) = \int_{-a}^{+a}\left(\int_{-a}^{+a}\left(\int_{-a}^{+a}(x^2+y^2)dx\right)dy\right)dz = \frac{M(2a)^2}{6} = \frac{\rho 2^5 a^5}{6}$$

$$I_{z'}(B) = \rho\int_0^{\alpha}\sin^3\theta d\theta\int_0^{2\pi}d\varphi\int_0^R r^4 dr = \rho 2\pi\frac{R^5}{5}\frac{(1-\cos\alpha)}{3}(2-\cos\alpha-\cos^2\alpha) = \rho\frac{2\pi}{3}\frac{R^5}{5}(2-3\cos\alpha+\cos^3\alpha)$$

$$I_{z'}(C) = \rho\int_0^{2\pi}d\varphi\int_0^{R\cos\alpha}\left(\int_0^{z.tg\alpha}r^3 dr\right)dz = \rho 2\pi\frac{R^5}{5}\frac{tg^4\alpha}{4}.\cos^5\alpha = \frac{\rho\pi R^5\sin^4\alpha.\cos\alpha}{10}$$

$$\Rightarrow I_z = I_z(A) - I_z(B) + I_z(C) = \frac{\rho 16a^5}{3} - \rho\frac{4\pi}{15}R^5 + \frac{\rho\pi R^5}{2}\cos\alpha - \frac{\rho\pi R^5}{3}\cos^3\alpha + \frac{\rho\pi R^5\cos^5\alpha}{10}$$



4.2

Si $R = a, \alpha = \frac{\pi}{2}$ $(\cos\alpha = 0): I_z = \frac{\rho a^5}{15}(80-4\pi)$

$$I_z(sphère) = \frac{2}{5}MR^2 = \frac{8\rho\pi R^5}{15} \quad \text{et} \quad I_z(\frac{1}{2}sphère) = \frac{2}{5}M'R^2 = \frac{4\rho\pi R^5}{15}$$

4.3 $\boxed{\begin{array}{l}\text{Solide possédant un axe de révolution suivant z}\\ I_z = \int\left(x^2+y^2\right)dm \text{ et } I_x = I_y \text{ par symétrie. Donc : } I_x + I_y = I_z + 2\int z^2 dm \Rightarrow I_x = \frac{I_z}{2} + \int z^2 dm\end{array}}$

demi-sphère :

$$I_{x'}(\text{demi-sphère} \cup) = \frac{2\rho\pi R^5}{15} + \rho\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^R \left(r\cos\theta\right)^2 dr.rd\theta.r\sin\theta d\varphi = \frac{2\rho\pi R^5}{15} + \frac{2\rho\pi R^5}{15} = \frac{4\rho\pi R^5}{15}$$

$$I_{x'}(\cup) = I_{y'}(\cup)$$

$$I_x(\cup) = \underbrace{I_{x'}(\cup) - M_\cup\left(\frac{3}{8}R\right)^2}_{I_{xG}(\cup)} + M_\cup\left(\frac{5}{8}R\right)^2 = \frac{13M_\cup R^2}{20} = \frac{13\rho\pi R^5}{30}$$

Solide complet :

$$I_x = I_x(\text{Cube}) - I_x(\cup) = \frac{16\rho R^5}{3} - \frac{13\rho\pi R^5}{30} = \frac{\rho R^5}{30}\left(160-13\pi\right) \text{ et } I_z = \frac{\rho R^5}{30}\left(160-8\pi\right) \Rightarrow I_y = I_x < I_z$$

Pour que l'ellipsoïde d'inertie soit une sphère, il faut trouver un point P tel que :

$$I_x x^2 + I_y y^2 + I_z z^2 = 1 \text{ où } I_x = I_y = I_z$$

$$\Rightarrow \underbrace{I_x - \left(m_{Carré} - m_\cup\right)z_G^2}_{I_{xG_{total}}} + \left(m_{Carré} - m_\cup\right)d^2 = I_z$$ où d est la distance du point P recherché à G_{total}

---

5. Pour la demi sphère pleine.

$$M_1 = \frac{\rho 2\pi r^3}{3} \text{ et } z'_{G1} = \frac{5r}{8}.$$

$$I_{z'} = \frac{2M_1 r^2}{5}$$

Axes $x_{O'}y_{O'}z'$ en O'

$$I_{x_{O'}} = I_{x_{O'}y_{O'}} + \frac{I_{z'}}{2} = \frac{M_1 r^2}{5} + \frac{M_1 r^2}{5} = \frac{2M_1 r^2}{5}$$

$$I_{x'} = I_{x_{G1}} + M_1 d_{x'x_{G1}}{}^2 = \left(I_{x_{O'}} - M_1 d_{x_{O'}x_{G1}}{}^2\right) + M_1 d_{x'x_{G1}}{}^2 = M_1\left(\frac{2r^2}{5} - \frac{9r^2}{64} + \frac{25r^2}{64}\right) = \frac{13M_1 r^2}{20}$$

Pour le cylindre plein.

$$M_2 = \rho\pi L r^2 \text{ et } z'_{G2} = r + \frac{L}{2}$$

$$I_{z'} = \frac{M_2 r^2}{2}$$

$$I_{x_{G2}} = I_{x_{G2}y_{G2}} + \frac{I_{z'}}{2} = \frac{M_2 L^2}{12} + \frac{M_2 r^2}{4}$$

$$I_{x'} = I_{x_{G2}} + M_2 d_{x'x_{G2}}{}^2 = \frac{M_2 L^2}{12} + \frac{M_2 r^2}{4} + M_2\left(\frac{L}{2}+r\right)^2 = M_2\left(\frac{L^2}{12} + \frac{r^2}{4} + \frac{L^2}{4} + r^2 + Lr\right) = M_2\left(\frac{L^2}{3} + \frac{5r^2}{4} + Lr\right)$$

Pour le cône.

$$M_3 = \rho\pi 2R\sqrt{4R^2+4H^2} - \rho\pi R\sqrt{R^2+H^2} = 3\rho\pi R\sqrt{R^2+H^2}$$

$$I_{z'} = \frac{\rho\pi 2R\sqrt{4R^2+4H^2}\,4R^2}{2} - \frac{\rho\pi R\sqrt{R^2+H^2}\,R^2}{2} = \frac{15\rho\pi R\sqrt{R^2+H^2}}{2} = \frac{5M_3 R^2}{2}$$

A la pointe du cone ($x_p y_p z'$) :

$$I_{x_P} = \frac{I_{z_P}}{2} + I_{x_P y_P} = \frac{5M_3 R^2}{4} + \left[\frac{\rho\pi 2R\sqrt{4R^2+4H^2}\,4H^2}{2} - \frac{\rho\pi R\sqrt{R^2+H^2}\,H^2}{2}\right] = \frac{5M_3 R^2}{4} + \frac{5M_3 H^2}{2}$$

$$I_{x'} = I_{x_{G3}} + M_3 d_{x_{G3}x'}{}^2 = \left(I_{x_P} - M_3 d_{x_P x_{G3}}{}^2\right) + M_3 d_{x_{G3}x'}{}^2$$

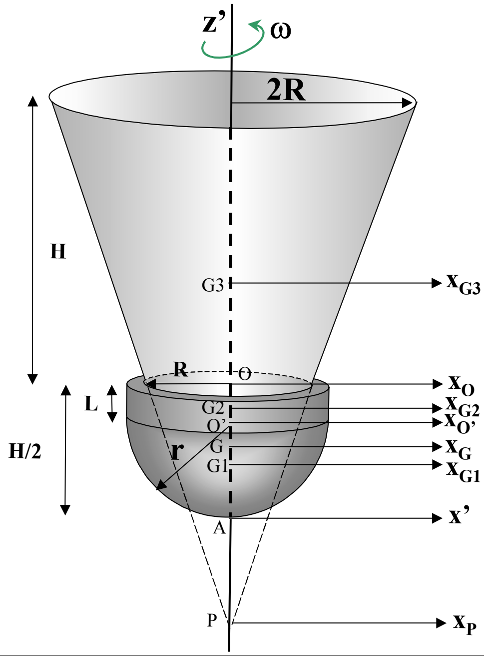
$$= \frac{5M_3 R^2}{4} + \frac{5M_3 H^2}{2} - M_3\left(\frac{14H}{9}\right)^2 + M_3\left(\frac{19H}{18}\right)^2 = \frac{5M_3 R^2}{4} + \frac{43M_3 H^2}{36}$$

Pour le volant complet :

$$M = M_1 + M_2 + M_3 = \frac{\rho_1 2\pi r^3}{3} + \rho_2 \pi L r^2 + 3\rho_3 \pi R\sqrt{R^2 + H^2}$$

$$I_{z'} = I_{z'Sphère} + I_{z'Cylindre} + I_{z'Cone} = \frac{2M_1 r^2}{5} + \frac{M_2 r^2}{2} + \frac{5M_3 R^2}{2}$$

$$I_{x'} = I_{y'} = I_{x'Sphère} + I_{x'Cylindre} + I_{x'Cone} = \frac{13M_1 r^2}{20} + M_2\left(\frac{L^2}{3} + \frac{5r^2}{4} + Lr\right) + \frac{5M_3 R^2}{4} + \frac{43M_3 H^2}{36}$$



Pour les problèmes relatifs au Tps et aux laboratoires, contactez Emmanuelle.Vin@ulb.ac.be
Pour les problèmes relatifs aux projets Matlab, contactez CFAO.Matlab@ulb.ac.be
http://cfao.ulb.ac.be/cfao/ >Teaching>mécaII>Tps. Login : ***student***, mot de passe : ***newton***

Attention : Pour les séries du mardi, la séance 6 sera donnée le mardi 16/12 comme indiqué aux valves.
La matière de l'examen portera sur la totalité des 6 premières séances des TPs.