1. $\sum F = ma$

$$N = mg\cos\theta$$

$$ma = m\frac{dv}{dt} = mg\sin\theta - 0.3mg\cos\theta$$

$$m\frac{dv^2}{2dl} = mg\sin\theta - 0.3mg\cos\theta$$

$$m\frac{1}{2}(v_2{}^2 - v_1{}^2) = (mg\sin\theta - 0.3mg\cos\theta)(l_2 - l_1)$$

$$m\frac{1}{2}(0.14^2 - 0.4^2) = (mg\sin\theta - 0.3mg\cos\theta)\frac{1.5}{\sin\theta}$$

$$-0.0702 = (9.81)(1.5)\left(1 - \frac{0.3}{tg\theta}\right)$$

$$\mathrm{tg}\theta = \left(\frac{0.0702 + (9.81)(1.5)}{(0.3)(1.5)(9.81)}\right)^{-1} = 0.299 \;\Rightarrow\; \theta = 16.62°$$

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2. $\bar{\omega}\big|_{Oxyz} = -p\cos\theta\,\bar{1}_x + \dot{\theta}\,\bar{1}_y + (p\sin\theta + \Omega)\,\bar{1}_z$

$$\bar{\varepsilon}\big|_{Oxyz} = \left.\frac{d\bar{\omega}}{dt}\right|_{rel} + \bar{\Omega}\times\bar{\omega} = \dot{\theta}(p\sin\theta - \Omega)\,\bar{1}_x - p\cos\theta\Omega\,\bar{1}_y + p\cos\theta\dot{\theta}\,\bar{1}_z$$

$$\bar{v}_G\big|_{Oxyz} = \bar{v}_{O'} + \bar{\omega}\times\bar{O}'\bar{G} = (h\sin\theta\dot{\theta})\,\bar{1}_x + \Omega(L - h\cos\theta)\,\bar{1}_y + (h\cos\theta\dot{\theta})\,\bar{1}_z$$

$$\bar{j}_G\big|_{Oxyz} = \left.\frac{d\bar{v}_G}{dt}\right|_{rel} + \bar{\Omega}\times\bar{v}_G = (h\cos\theta\dot{\theta}^2)\,\bar{1}_x + (\Omega h\sin\theta\dot{\theta})\,\bar{1}_y + (-h\sin\theta\dot{\theta}^2)\,\bar{1}_z + h\sin\theta\dot{\theta}\Omega\,\bar{1}_y - \Omega(L - h\cos\theta)\Omega\,\bar{1}_x$$

$$= \left(h\cos\theta(\dot{\theta}^2 + \Omega^2) - \Omega^2 L\right)\bar{1}_x + 2\Omega h\sin\theta\dot{\theta}\,\bar{1}_y - h\sin\theta\dot{\theta}^2\,\bar{1}_z$$

$$\bar{v}_G\big|_{OXYZ} = \left(h\sin\theta\cos\alpha\dot{\theta} - \Omega\sin\alpha(L - h\cos\theta)\right)\bar{1}_X + \left((R - h\cos\theta)\cos\alpha\Omega + h\sin\theta\sin\alpha\dot{\theta}\right)\bar{1}_Y + (h\cos\theta\dot{\theta})\,\bar{1}_Z$$

$$\bar{j}_G\big|_{OXYZ} = \left(\left(h\cos\theta(\dot{\theta}^2 + \Omega^2) - \Omega^2 R\right)\cos\alpha - 2h\sin\alpha\sin\theta\dot{\theta}\Omega\right)\bar{1}_X + \left(2h\sin\theta\cos\alpha\dot{\theta}\Omega + \left(h\cos\theta(\dot{\theta}^2 + \Omega^2) - \Omega^2 R\right)\sin\alpha\right)\bar{1}_Y$$
$$- h\sin\theta\dot{\theta}^2\,\bar{1}_Z$$

3.1 Centre de masse du volant G : 9r/10 à partir de A avec r=3R/2. (r/3+r=2R). Moment cinétique en A :

$$\bar{m}_A = M\;\bar{AG}\times\bar{v}_A + \bar{I}_A.\bar{\omega} = M\left(\frac{9r}{10}\bar{1}_{z'}\right)\times\left(-v_G\,\bar{1}_{z'} - \frac{9r}{10}\omega_2\,\bar{1}_{x'}\right) + \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix}.\begin{pmatrix} 0 \\ \omega_2\,\bar{1}_{y'} \\ \omega_1\,\bar{1}_{z'} \end{pmatrix}$$

avec $\bar{v}_A = \bar{v}_G + \bar{\omega}\times\bar{GA} = -v_G\,\bar{1}_{z'} + (\omega_2\,\bar{1}_{y'} + \omega_1\,\bar{1}_{z'})\times\left(-\frac{9r}{10}\bar{1}_{z'}\right) = -v_G\,\bar{1}_{z'} - \frac{9r}{10}\omega_2\,\bar{1}_{x'}$

$$\bar{m}_A = -M\left(\frac{9r}{10}\right)^2\omega_2\,\bar{1}_{y'} + (A\omega_2\,\bar{1}_{y'} + C\omega_1\,\bar{1}_{z'}) = \left(A\omega_2 - M\left(\frac{9r}{10}\right)^2\omega_2\right)\bar{1}_{y'} + C\omega_1\,\bar{1}_{z'}$$

3.2 L'énergie cinétique :

$$T = \frac{M}{2}\bar{v}_G{}^2 + \frac{1}{2}\bar{\omega}.\bar{I}_G.\bar{\omega} = \frac{M}{2}v_G{}^2 + \frac{1}{2}\begin{pmatrix} 0 & \omega_2 & \omega_1 \end{pmatrix}.\begin{pmatrix} A - M(9r/10)^2 & 0 & 0 \\ 0 & A - M(9r/10)^2 & 0 \\ 0 & 0 & C \end{pmatrix}.\begin{pmatrix} 0 \\ \omega_2 \\ \omega_1 \end{pmatrix}$$

$$= \frac{M}{2}\left(v_G{}^2 - \left(\frac{9r}{10}\right)^2\omega_2{}^2\right) + \frac{1}{2}(A\omega_2{}^2 + C\omega_1{}^2)$$

$$T = \frac{M}{2}\bar{v}_A{}^2 + M\;\bar{v}_A.(\bar{\omega}\times\bar{AG}) + \frac{1}{2}\bar{\omega}.\bar{I}_A.\bar{\omega}$$

$$\frac{M}{2}\left(v_G{}^2 + \left(\frac{9r}{10}\right)^2\omega_2{}^2\right) + M\left(-v_G\,\bar{1}_{z'} - \frac{9r}{10}\omega_2\,\bar{1}_{x'}\right).\left((\omega_2\,\bar{1}_{y'} + \omega_1\,\bar{1}_{z'})\times\left(\frac{9r}{10}\bar{1}_{z'}\right)\right) + \frac{1}{2}\begin{pmatrix} 0 & \omega_2 & \omega_1 \end{pmatrix}.\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix}.\begin{pmatrix} 0 \\ \omega_2 \\ \omega_1 \end{pmatrix}$$

$$\frac{M}{2}\left(v_G{}^2 + \left(\frac{9r}{10}\right)^2\omega_2{}^2\right) + M\left(-v_G\,\bar{1}_{z'} - \frac{9r}{10}\omega_2\,\bar{1}_{x'}\right).\left(\frac{9r}{10}\omega_2\,\bar{1}_{x'}\right) + \frac{1}{2}(A\omega_2{}^2 + C\omega_1{}^2)$$

$$T = \frac{M}{2}\left(v_G{}^2 + \left(\frac{9r}{10}\right)^2\omega_2{}^2\right) + M\left(-\omega_2{}^2\left(\frac{9r}{10}\right)^2\right) + \frac{1}{2}(A\omega_2{}^2 + C\omega_1{}^2) = \frac{M}{2}\left(v_G{}^2 - \left(\frac{9r}{10}\right)^2\omega_2{}^2\right) + \frac{1}{2}(A\omega_2{}^2 + C\omega_1{}^2)$$

3.3 Pour la demi sphère pleine.

$$M_1 = \frac{\rho 2\pi r^3}{3} \text{ et } z'_{G1} = \frac{5r}{8}.$$

$$I_{z'} = \frac{2M_1 r^2}{5}$$

Axes $x_{O'} y_{O'} z'$ en O'

$$I_{x_{O'}} = I_{x_{O'} y_{O'}} + \frac{I_{z'}}{2} = \frac{M_1 r^2}{5} + \frac{M_1 r^2}{5} = \frac{2M_1 r^2}{5}$$

$$I_{x'} = I_{x_{G1}} + M_1 d_{x' x_{G1}}{}^2 = \left(I_{x_{O'}} - M_1 d_{x_{O'} x_{G1}}{}^2\right) + M_1 d_{x' x_{G1}}{}^2 = M_1\left(\frac{2r^2}{5} - \frac{9r^2}{64} + \frac{25r^2}{64}\right) = \frac{3M_1 r^2}{20}$$

Pour le cylindre plein.

$$M_2 = \rho\pi L r^2 \text{ et } z'_{G2} = r + \frac{L}{2}$$

$$I_{z'} = \frac{M_2 r^2}{2}$$

$$I_{x_{G2}} = I_{x_{G2} y_{G2}} + \frac{I_{z'}}{2} = \frac{M_2 L^2}{12} + \frac{M_2 r^2}{4}$$

$$I_{x'} = I_{x_{G2}} + M_2 d_{x' x_{G2}}{}^2 = \frac{M_2 L^2}{12} + \frac{M_2 r^2}{4} + M_2\left(\frac{L}{2} + r\right)^2 = M_2\left(\frac{L^2}{12} + \frac{r^2}{4} + \frac{L^2}{4} + r^2 + Lr\right) = M_2\left(\frac{L^2}{3} + \frac{5r^2}{4} + Lr\right)$$

Pour le cône.

$$M_3 = \rho\pi 2R\sqrt{4R^2 + 4H^2} - \rho\pi R\sqrt{R^2 + H^2} = 3\rho\pi R\sqrt{R^2 + H^2}$$

$$I_{z'} = \frac{\rho\pi 2R\sqrt{4R^2 + 4H^2}\,4R^2}{2} - \frac{\rho\pi R\sqrt{R^2 + H^2}\,R^2}{2} = \frac{15\rho\pi R\sqrt{R^2 + H^2}}{2} = \frac{5M_3 R^2}{2}$$

A la pointe du cone ($x_p y_p z'$) :

$$I_{x_p} = \frac{I_{z_P}}{2} + I_{x_P y_P} = \frac{5M_3 R^2}{4} + \left[\frac{\rho\pi 2R\sqrt{4R^2 + 4H^2}\,4H^2}{2} - \frac{\rho\pi R\sqrt{R^2 + H^2}\,H^2}{2}\right] = \frac{5M_3 R^2}{4} + \frac{5M_3 H^2}{2}$$

$$I_{x'} = I_{x_{G3}} + M_3 d_{x_{G3} x'}{}^2 = \left(I_{x_P} - M_3 d_{x_P x_{G3}}{}^2\right) + M_3 d_{x_{G3} x'}{}^2$$

$$= \frac{5M_3 R^2}{4} + \frac{5M_3 H^2}{2} - M_3\left(\frac{14H}{9}\right)^2 + M_3\left(\frac{19H}{18}\right)^2 = \frac{5M_3 R^2}{4} + \frac{43M_3 H^2}{36}$$

Pour le volant complet :

$$M = M_1 + M_2 + M_3 = \frac{\rho_1 2\pi r^3}{3} + \rho_2 \pi L r^2 + 3\rho_3 \pi R\sqrt{R^2 + H^2}$$

$$I_{z'} = I_{z'Sphère} + I_{z'Cylindre} + I_{z'Cone} = \frac{2M_1 r^2}{5} + \frac{M_2 r^2}{2} + \frac{5M_3 R^2}{2}$$

$$I_{x'} = I_{y'} = I_{x'Sphère} + I_{x'Cylindre} + I_{x'Cone} = \frac{3M_1 r^2}{20} + M_2\left(\frac{L^2}{3} + \frac{5r^2}{4} + Lr\right) + \frac{5M_3 R^2}{4} + \frac{43M_3 H^2}{36}$$