EQUATIONS DE HAMILTON (1)

- Soit un système soumis à des forces dérivant d'un potentiel :

$$L(q_i, \dot{q}_i, t) = T - V \qquad i = 1, \dots, n$$

- Introduisons la fonction Lagrangienne adjointe L^* :

$$\boxed{L^*(q_i, \dot{q}_i, t) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L}$$

- Introduisons les momentoïdes conjugués p_i aux q_i :

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}(q_j, \dot{q}_j, t)} \qquad \Rightarrow \qquad \text{par inversion : } \dot{q}_i(q_j, p_j, t)$$

- Définissons la fonction de Hamilton $H(q_i, p_i, t)$:

$$\boxed{H(q_i, p_i, t) \equiv L^*\left[q_i, \dot{q}_i(q_j, p_j, t), t\right]}$$

$$dH = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt \qquad (1)$$

EQUATIONS DE HAMILTON (2)

$$dH = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt \qquad (1)$$

$$\ldots q_i, \dot{q}_i, t) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$$dL^* = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \sum_{i=1}^{n} \dot{q}_i d\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt \qquad ($$

$$dL^* = \sum_{i=1}^{n} \dot{q}_i d\, p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt$$

➔ **En comparant (1) et (2) :**

$$\left.\begin{aligned} \frac{\partial H}{\partial q_i} &= -\frac{\partial L}{\partial q_i} \\ \frac{\partial H}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \end{aligned}\right\} \quad i = 1,\ldots,n$$

➔ **En utilisant les équations de Lagrange et la définition des momentoïdes conjugués, nous obtenons :**

$$\boxed{\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}} \qquad i = 1,\ldots,n$$

POINT MATERIEL DANS UN CHAMP DE POTENTIEL (1)

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad L = T - V$$

➔ **En coordonnées cartésiennes :** $$p_i = \frac{\partial T}{\partial \dot{q}_i}$$

$$\begin{cases} p_x = m\dot{x} \\ p_y = m\dot{y} \\ p_z = m\dot{z} \end{cases}$$

$$H = p_i\dot{q}_i - L = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - T + V$$

$$= \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V$$

$$\boxed{H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V}$$

POINT MATERIEL DANS UN CHAMP DE POTENTIEL (2)

➔ **Equations de Hamilton :**

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

$$\dot{x} = \frac{p_x}{m} \qquad \frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}$$

$$m\ddot{x} = -\frac{\partial V}{\partial x}$$