MATH-H-405 - Decision engineering

Session 1: Voting theory

Exercise 1 - The plurality voting
Demonstrate or give a counterexample:

1. Is the plurality voting method monotonic?
2. Is the plurality voting method independent to third alternatives?
3. Is the plurality voting method separable?

Exercise 2 - Condorcet procedure
Demonstrate or give a counterexample:

1. Is the Condorcet method monotonic?
2. Is the Condorcet method separable?

Exercise 3 - Arrow's theorem
Following the hypotheses of Arrow's theorem, demonstrate:

1. $\forall x, y \in A$, xSy or yPx.
2. If xPy and $ySz \Rightarrow xPz$.
3. If there exists x' and y' such that we have $x'Py'$ each time that $x'P_1y'$ and $y'P_2x'$. Then each time that $x'P_1y'$, we have $x'Py'$.

Exercise 4
In a group of ten persons that have defined a total order for a set A of candidates, we take as a collective decision rule that *"a is better than b"* if at least 6 persons out of 10 rank a before b.

1. Is the given relationship *"is better than"* always transitive?
2. Can this relationship have cycles?

Same questions for the relationship *"a is better than b"* if at least 7 persons out of 10 rank a before b.

Exercise 5 - Borda
We consider the nomination of a manager in a group of m persons. We suppose that each

voter has established a total order for the m persons. Firstly, we choose n candidates ($n < m$) that have earned the highest Borda score (in case of *ex aequo*, we choose the candidate at random). Secondly, we proceed to a second vote with a Borda procedure where only the n selected persons vote.
Can we end with the nomination of a manager such that a majority of voters still prefer another candidate? Demonstrate or give a counterexample.

Exercise 6 - Jefferson
By applying Jefferson's rule, demonstrate:

$$p_i < p_j \Rightarrow s_i \leq s_j.$$

Exercise 7 - Banzhaf
The Board of Supervisors in Nassau County, New York uses a weighted voting system that gives representation to each of six districts in the county according to their relative populations. In 1965, a total of 115 votes were allocated to the districts, as shown in the table below. For a motion to be passed, a majority of the total number of votes was required, and the the quota for the system was 58.

District	Weight
Hempstead #1	31
Hempstead #2	31
Oyster Bay	28
North Hempstead	21
Long Beach	2
Glen Cove	2

Table 1: Nassau County Board of Supervisors, 1965

1. In a series of lawsuits, Banzhaf successfully argued that all the power within the board was equally distributed among the three largest districts. Without actually calculating the Banzhaf power of index of any of the districts, explain why this was in fact the case.

2. Make a list of all the winning coalitions for the system.

3. For each of the winning coalitions from point 2., identify all of the voters that are critical to the coalition.

4. Using your answer of point 3., determine the Banzhaf index of each of the six districts in the system.

Reference: *"The Mathematics of Voting and Election: A Hands-On Approach"*, J.K. Hodge, R.E. Klima