MATH-H-405 - Decision engineering

Solutions of Session 1: Voting theory

NOTATION:

- The set $A = \{a_1, \dots a_k\}$ of candidates.
- The set $V = \{1, 2, \dots N\}$ of voters.
- For each voter $n \in V$ and for each candidate $a_i \in A$, we note $R_n(a_i)$ the position of the candidate a_i in the ranking of the voter n.
- Let us note $S(a_i)$ the score of the plurality voting earned by the candidate a_i.
- Let us note n_{ij} the Condorcet score of the action a_i on the action a_j.
- Let us note $B(a)$ the Borda score earned by the candidate a.

Exercise 1

1. Suppose the candidate $a_i \in A$ improve his position in the ranking R_n $(n \in V)$. Let us note R'_n the new order of the candidate for the voter n and S' the new scores of the plurality voting. There are two cases:

   - $R'_n(a_i) = 1$. In this case, let us note a_j $(j = 1, \dots, k$ and $i \neq j)$, the candidate such that $R'_n(a_j) = 2$. We have thus:

   $$\begin{aligned} S'(a_i) &= S(a_i) + 1 \\ S'(a_j) &= S(a_j) - 1 \\ S'(a_s) &= S(a_s)\ \forall s \neq i, j \end{aligned}$$

   - $R'_n(a_i) \neq 1$

   $$S'(a_s) = S(a_s)\ \forall s = 1, \dots, k$$

   The plurality voting is thus monotonic.

2. The plurality voting is not independent to third alternatives. Indeed let us consider the following example:
   Example
   Let us consider the following voting strategy:

- *4 persons vote $a > b > c$*
- *2 persons vote $c > b > a$*
- *3 persons vote $b > c > a$.*

The candidate a is thus elected. If c withdraws, b is elected.

3. Let us consider $V = B \cup \bar{B}$ and note S_B and $S_{\bar{B}}$ the scores of the plurality vote respectively given by the regions B and $\bar{B}$. In those two regions, we have the same ranking. We will suppose, without any restrictions, that:

$$S_B(a_1) \geq S_B(a_2) \geq \cdots \geq S_B(a_k)$$

$$S_{\bar{B}}(a_1) \geq S_{\bar{B}}(a_2) \geq \cdots \geq S_{\bar{B}}(a_k)$$

Since $S(a_i) = S_B(a_i) + S_{\bar{B}}(a_i)\ \forall i = 1, \ldots, k$, the ranking of the whole region will be the same as the one given by the two separate regions B and $\bar{B}$.

Exercise 2

1. We have to consider two cases for the monotony:

   (a) If the candidate is elected and improves its score, it will remain elected. Indeed, the Condorcet winner is a candidate who opposed to any other candidate is always the winner. Let a be the Condorcet winner. We have thus $\forall j \neq a$: $n_{aj} > n_{ja}$. If a improves its ranking for m voters to the detriment of a candidate $i \neq a$, $n_{ai} > n_{ia}$ (being the Condorcet winner, a was already prefered to i) becomes $n_{ai} + m > n_{ia} - m$, which increases the preference. Since we still have $\forall j \neq a, i : n_{aj} > n_{ja}$, a remains elected.

   (b) If the candidate is not elected but improves its ranking for a voter, it will not necessarily improves its ranking in the global ranking. Condorcet is thus not monotonic in that sense. Indeed, let us consider the following example:
   Example
   Let us consider the following voting strategy:

   - *1 person votes $a > b > c$*
   - *1 person votes $b > c > a$*
   - *1 person votes $a > c > b$.*

   After applying the Condorcet method, we have the global ranking: $a > b > c$. If c improves his position in the third ranking so that we have $c > a > b$ for the third voter, there will be no solution for the Condorcet method (we have a cycle).

2. Let us consider that $V = B \cup \bar{B}$ and note n_{ij}^{B} and $n_{ij}^{\bar{B}}$ the Condorcet scores of a_i on a_j, respectively given by the regions B and $\bar{B}$. In those two regions, we have the same ranking. Let us suppose that a_i is globally preferred to a_j in the region B and in the region $\bar{B}$. Since $n_{ij} = n_{ij}^{B} + n_{ij}^{\bar{B}}$, a_i will be preferred to a_j in the global ranking when we consider V entirely.

Exercise 3
Following the hypotheses of Arrow's theorem, let us show that:

1. $\forall x, y \in A$, xSy or yPx
   We can demonstrate it *ad absurdum*. The negation of the affirmation is $\exists x, y \in A$, ySx AND xPy, but since we have the hypothesis xSy, this is a contradiction.

2. If xPy and $ySz \Rightarrow xPz$.
   Since xSy and ySz, we can apply the second axiom (transitivity of S) so we have xSz.

   Let us suppose *ad absurdum* that xIz. We also have that zSx(*ad absurdum*). Since the hypothesis is ySz, we have (by applying the transitivity of S) that ySx which will lead to yPx which is a contradiction of the hypothesis xPy.

3. Suppose x' and y' that respect the hypotheses of this proposition. Let us consider a ranking R_1 where $x'P_1y'$ and any other ranking R_2. What happens if we have xPy in the global ranking built with R_1 and R_2.

   From R_1 and R_2, we build two new rankings R_1' and R_2' such that $R_1' = R_1$ and R_2' is given by placing x' in the last position of R_2.

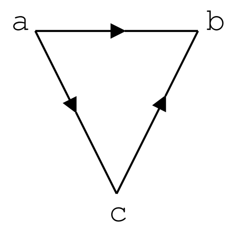
   When we consider those two new rankings R_1' and R_2', we know, following the hypothesis, that $x'Py'$ in the global ranking R'.

   With the monotony property (since x' can only improves or remains at the same position in R_2 in comparison with R_2'), we can conclude that $x'Py'$.

Exercise 4

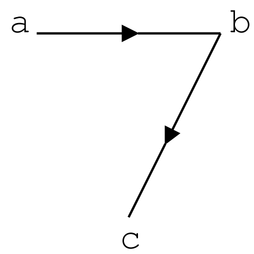
1. The first relationship is not always transitive and cycles can appear. This is illustrated by the following example:
   Example
   Let us consider the following voting strategy:

   - *3 persons vote* $a > b > c$
   - *3 persons vote* $c > a > b$
   - *4 persons vote* $b > c > a$.



2. The second relationship is not transitive as well, as shown by the following example:
   Example
   Let us consider the following voting strategy:

   - *4 persons vote $a > b > c$*
   - *3 persons vote $c > a > b$*
   - *3 persons vote $b > c > a$.*



However, this relationship does not have any cycles. Let us demonstrate it in the case of 3 candidates: Let us suppose that R_1 persons vote $a > b > c$, R_2 $a > c > b$, R_3 $b > a > c$, R_4 $b > c > a$, R_5 $c > a > b$ et R_6 $c > b > a$ (which are all the permutations for the three candidates). To find a cycle, there should be:

$$R_1 + R_2 + R_5 \geq 7(a > b) \quad (1)$$
$$R_1 + R_3 + R_4 \geq 7(b > c) \quad (2)$$
$$R_4 + R_5 + R_6 \geq 7(c > a). \quad (3)$$
$$\quad (4)$$

We also have that:

$$R_1 + R_2 + R_3 + R_4 + R_5 + R_6 = 10. \quad (5)$$

By adding up (1),(2) et (3), we have:

$$2R_1 + R_2 + R_3 + R_4 + 2R_5 + R_6 \geq 21. \quad (6)$$
$$\quad (7)$$

Using 5, we find the following contradiction:

$$R_1 + R_4 + R_5 \geq 11.$$

Exercise 5
This process does not respect the Condorcet criterion as illustrated below:
Example
Let us suppose that $m = 3$ and $n = 2$ and the following voting strategy:

- *a votes $a > c > b$*
- *b votes $b > a > c$*
- *c votes $b > a > c$*

We find that $B(a) = 7, B(b) = 7$ and $B(c) = 4$. On the first round, a and b will be chosen. If we apply Borda while only considering the vote of a and b (and keeping the same Borda score as previously), we find $B(a) = 5$ and $B(b) = 4$ so a is thus elected. However, b and c prefer b to a.

Exercise 6
The Jefferson rule implies that:

$$\frac{p_i}{s_i} \geq \frac{p_j}{s_j + 1}$$

This can be rewritten as follows:

$$\frac{p_i}{p_j} \geq \frac{s_i}{s_j + 1}$$

However $\frac{p_i}{p_j} < 1$, so $\frac{s_i}{s_j+1} < 1$.
We then have $s_i < s_j + 1$ and thus $s_i \leq s_j$.

Exercise 7

1. Take the three largest districts two at a time along with none of the three smallest districts, and see what the weights of the resulting coalitions would be. Then take the three largest districts one at a time along with all three of the smallest districts, and see what the weights of the resulting coalitions would be.
2. There are a total of 32 winning coalitions for the system. Of these, 3 have two members, 10 have three members, 12 have four members, 6 have five members, and 1 has six members.
   For example: {Hempstead #1, Hempstead #2}, {Hempstead #1, Oyster Bay}, {Hempstead #2, Oyster Bay}, {Hempstead #1, Hempstead #2, Oyster Bay}, etc.

3. There are a total of 48 critical voters in all of the winning coalitions for the system.
4. Computed from point 1. 2. and 3.