MATH-H-405 - Decision engineering

Solutions of Session 2&3: Multi-criteria decision aid

Exercice 1

$aPb, bPc, aId \Rightarrow dPc$

$$\begin{aligned} aPb &: g(a) > g(b) + q & (1)\\ bPc &: g(b) > g(c) + q & (2)\\ aId &: -q + g(d) \leq g(a) \leq q + g(d) & (3) \end{aligned}$$

(1) and (2) :

$$g(a) > g(c) + 2q \tag{4}$$

(3) (right member) and (4) :

$$g(d) > g(c) + q \tag{5}$$

$aPb, bIc, cPd \Rightarrow aPd$

$$\begin{aligned} aPb &: g(a) > g(b) + q & (6)\\ bIc &: -q + g(c) \leq g(b) \leq q + g(c) & (7)\\ aId &: g(c) > g(d) + q & (8) \end{aligned}$$

(6) and (7) (left member) :

$$g(a) > g(c) \tag{9}$$

(8) and (9) :

$$g(a) > g(d) + q \tag{10}$$

Exercise 2

The solution space is the green $[A, B]$ segment in Fig. 1.

In the criteria space, we see that the set of efficient points can be described as follows:

$$\{\lambda(5;1) + (1-\lambda)(4;2) | \lambda \in [0,1]\} = \{(\lambda + 4; 2 - \lambda) | \lambda \in [0,1]\}$$

The Datum point algorithm consists in finding the point which is the nearest to the ideal point $(5, 2)$ following a L^1 or L^2 distance.

Reminder:

- L^2 distance (euclidean distance) between the point $a(a_1, a_2)$ and the point $b(b_1, b_2)$: $d(a,b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$.

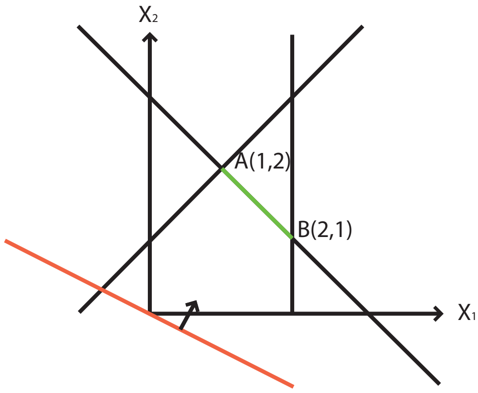


Figure 1: Solution space

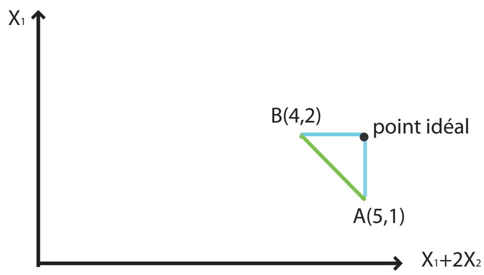


Figure 2: Criteria space

- L^1 distance (Manhattan distance) between the point $a(a_1, a_2)$ and the point $b(b_1, b_2)$: $d(a, b) = |a_1 - b_1| + |a_2 - b_2|$.

L^2 distance:
We have to find λ such that:

$$\arg\min_{\lambda}\{(5 - \lambda - 4)^2 + (2 - 2 + \lambda)^2\} = \arg\min_{\lambda}\{2\lambda^2 - 2\lambda + 1\}.$$

$$\Rightarrow \lambda = \frac{1}{2}$$

The point which is the nearest to the ideal point following a L^2 distance is thus $(4,5; 1,5)$. This result can be directly seen in Fig. 2
L^1 distance:
We have to find λ such that:

$$\arg\min_{\lambda}\{|5 - \lambda - 4| + |\lambda|\} = \arg\min_{\lambda}\{1\}$$

$\Rightarrow$ each $\lambda \in [0, 1]$ is solution and thus all the points of the $[A, B]$ segment are solution.

Exercise 3
Each point of a convex Pareto frontier can be found by maximizing a weighted sum where the weight are well-chosen. It will not be the case for a concave frontier.

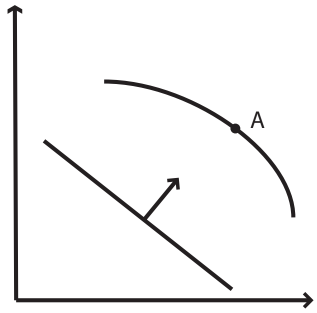


Figure 3: Criteria space: the point A can be found by maximizing a weighted sum

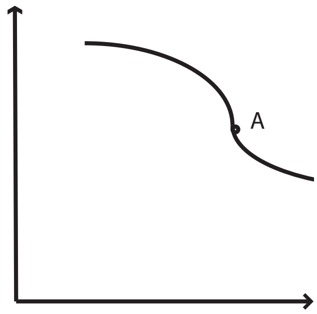


Figure 4: Criteria space: the point A can never be found by maximizing a weighted sum

Exercise 4
Remark that the given discordance threshold is not normalized ($\notin [0,1]$).
We do not need to divide by δ in the definition of the discordance threshold.
The kernel is $\{B\}$.

	A	B	C	D	E
A	1	0.5	0.15	0.4	0.4
B	0.75	1	0.35	0.75	0.6
C	0.85	0.65	1	0.6	0.5
D	0.6	0.25	0.4	1	0.25
E	0.6	0.6	0.75	0.75	1

Table 1: Concordance index

	A	B	C	D	E
A	0	1	1	1	1
B	0	0	0	1	0
C	1	1	0	0	0
D	0	1	1	0	1
E	0	1	0	1	0

Table 2: Discordance matrix (0 if discordance, 1 otherwise)

	A	B	C	D	E
A	1	0	0	0	0
B	1	1	0	0	1
C	0	0	1	1	0
D	1	0	0	1	0
E	1	0	1	0	1

Table 3: Outranking matrix (1 if outranking, 0 otherwise)

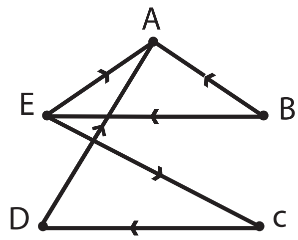


Figure 5: Outranking graph

Exercise 5
Preferential independence:
Let us remark that:

$$\begin{aligned} g_1(a_1) &= g_1(a_2) \\ g_1(a_4) &= g_1(a_5) \\ g_2(a_1) &= g_2(a_4) \\ g_2(a_2) &= g_2(a_5) \end{aligned}$$

The preferences of the decision maker verify that $a_4 P a_1 \Leftrightarrow a_5 P a_2$. We have to proceed that way for each quadruplet of actions that verify the preferential independence conditions.
Utility function:

$$a_3 I a_7 \Leftrightarrow U(a_3) = U(a_7)$$

We thus have:

$$U_1(1) + U_2(5) = U_1(3) + U_2(1). \tag{11}$$

Also,

$$a_2 I a_4 \Leftrightarrow U(a_2) = U(a_4).$$

Thus,

$$U_1(1) + U_2(3) = U_1(2) + U_2(1). \tag{12}$$

If we take (11)-(12)

$$\begin{aligned} U_2(5) - U_2(3) &= U_1(3) - U_1(2) && (13)\\ &\Updownarrow \\ U_1(2) + U_2(5) &= U_1(3) + U_2(3). && (14)\\ & && (15) \end{aligned}$$

In this case, $a_6 I a_8$. But it is not verified by the preferences of the decision maker since $a_6 P a_8$. It is thus not possible to model the preferences with a utility function.

Exercise 6
We can use a recursion method.
Let $N = 2$:

$$\phi(a_1) + \phi_(a_2) = \pi(a_1, a_2) - \pi(a_2, a_1) + \pi(a_2, a_1) - \pi(a_1, a_2) = 0.$$

Suppose that $\sum_{i=1}^{k} \phi(a_i) = 0$ and demonstrate that $\sum_{i=1}^{k+1} \phi(a_i) = 0$.

$$\begin{aligned} \sum_{i=1}^{k+1} \phi(a_i) &= \frac{1}{k} \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} \big(\pi(a_i, a_j) - \pi(a_j, a_i)\big) \\ &= \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{k+1} \big(\pi(a_i, a_j) - \pi(a_j, a_i)\big) + \frac{1}{k} \sum_{j=1}^{k+1} \big(\pi(a_{k+1}, a_j) - \pi(a_j, a_{k+1})\big) \\ &= \frac{k-1}{k} \frac{1}{k-1} \sum_{i=1}^{k} \sum_{j=1}^{k} \big(\pi(a_i, a_j) - \pi(a_j, a_i)\big) + \frac{1}{k} \sum_{i=1}^{k} \big(\pi(a_i, a_{k+1}) - \pi(a_{k+1}, a_i)\big) \\ &\quad + \frac{1}{k} \sum_{j=1}^{k} \big(\pi(a_{k+1}, a_j) - \pi(a_j, a_{k+1})\big) \\ &= \frac{k-1}{k} \sum_{i=1}^{k} \phi(a_i) \\ &= 0 \text{ (recursion hypothesis)} \end{aligned}$$

Exercise 7
PROMETHEE II verifies all the Arrow's condition except the independence to third alternatives. It is due to the fact that PROMETHEE II proceed with pairwise comparisons.

We will demonstrate that PROMETHEE II verifies the monotonicity condition (the verification of the three other properties is obvious).
Let $A = \{a_1, a_2, \ldots, a_n\}$ be the set of actions, $F = \{f_1, f_2, \ldots, f_m\}$ the criteria family. Let us consider $A' = \{a_1, a_2, \ldots, a'_i, \ldots, a_n\}$ where a'_i is defined such that:

$$\begin{array}{rcl} f_k(a'i) & > & f_k(a_i)\ k \in \{1, 2, \ldots, m\} \\ f_j(a'_i) & = & f_j(a_i)\ \forall j \in \{1, 2, \ldots, m\} \text{ and } j \neq k. \end{array}$$

Let us note $\phi'(a)$ the net flow of PROMETHEE for each action $a \in A'$.
We can remark that:

$$\begin{array}{rcl} \pi_j(a'_i, b) & = & \pi_j(a_i, b)\ \forall b \in A, \forall j \in \{1, 2, \ldots, m\} \text{ and } j \neq k \\ \pi_j(b, a'_i) & = & \pi_j(b, a_i)\ \forall b \in A, \forall j \in \{1, 2, \ldots, m\} \text{ and } j \neq k \\ \pi_k(a'_i, b) & \geq & \pi_k(a_i, b)\ \forall b \in A \\ \pi_k(b, a'_i) & \leq & \pi_k(b, a_i)\ \forall b \in A \end{array}$$

$$\begin{array}{rcl} \phi'(a'_i) & = & \frac{1}{n-1} \sum_{b \in A'} \big(\pi(a'_i, b) - \pi(b, a'_i)\big) \\ & = & \frac{1}{n-1} \sum_{b \in A'} \sum_{j=1}^{m} w_j \big(\pi_j(a'_i, b) - \pi_j(b, a'_i)\big) \\ & = & \frac{1}{n-1} \sum_{b \in A'} \sum_{j=1, j \neq k}^{m} w_j \big(\pi_j(a'_i, b) - \pi_j(b, a'_i)\big) + \frac{1}{n-1} w_k \big(\pi_k(a'_i, b) - \pi_k(b, a'_i)\big) \\ & \geq & \frac{1}{n-1} \sum_{b \in A} \sum_{j=1, j \neq k}^{m} w_j \big(\pi_j(a_i, b) - \pi_j(b, a_i)\big) + \frac{1}{n-1} w_k \big(\pi_k(a_i, b) - \pi_k(b, a_i)\big) \\ & = & \phi(a_i) \end{array}$$

We can demonstrate similarly that $\phi'(a) \leq \phi(a)\ \forall a \in A$ and $a \neq a_i$.

Exercise 8
Suppose that we remove an action $y \in A$. Let us note $\phi_y(a) = \frac{1}{n-2} \sum_{x \in A, x \neq y} \big(\pi(a, x) - \pi(x, a)\big)$ the net flow of the action $a \in A$ calculated when the action y is removed.

$$\begin{array}{rcl} \phi(a) & = & \frac{1}{n-1} \sum_{x \in A} \big(\pi(a, x) - \pi(x, a)\big) \\ & = & \frac{1}{n-1} \sum_{x \in A, x \neq y} \big(\pi(a, x) - \pi(x, a)\big) + \frac{1}{n-1} \big(\pi(a, y) - \pi(y, a)\big) \\ & = & \frac{n-2}{n-1} \phi_y(a) + \frac{1}{n-1} \big(\pi(a, y) - \pi(y, a)\big) \end{array}$$

We thus have:

$$\phi_y(a) \;=\; \frac{n-1}{n-2}\phi(a) - \frac{1}{n-2}\big(\pi(a,y) - \pi(y,a)\big) \tag{16}$$

Suppose that $\phi(a) - \phi(b) > 0$. We have, using (16), that

$$\phi_y(a) - \phi_y(b) \;=\; \frac{n-1}{n-2}\big(\phi(a) - \phi(b)\big) - \frac{1}{n-2}\big(\pi(a,y) - \pi(y,a) - \pi(b,y) + \pi(y,b)\big).$$

In order to avoid rank reversal, we must have that $\phi_y(a) - \phi_y(b) > 0$ which implies:

$$\phi(a) - \phi(b) \;>\; \frac{1}{n-1}\big(\pi(a,y) - \pi(y,a) - \pi(b,y) + \pi(y,b)\big)$$

However, since $\pi(c,d) \in [0,1] \forall c,d \in A$, we have

$$\big(\pi(a,y) - \pi(y,a) - \pi(b,y) + \pi(y,b)\big) \;<\; 2$$

We can conclude that $\phi_y(a) - \phi_y(b) > 0$ if

$$\phi(a) - \phi(b) > \frac{2}{n-1}$$

Exercise 9
Consider the following example:

Worker	Age	Degree	Professional experience
A	25	Master degree	No experience
B	25	No degree	3 years
C	35	Master degree	No experience
D	35	No degree	3 years

You would prefer worker A over worker B but worker D over worker C.

Exercise 10
The cohesion hypothesis says that if two alternatives, A and B, are equal in all criteria but one, and A is better than B according to that criterion, then A must be preferred to B. Now, if you consider a low level of certainty for that criterion, you can end up with wrong evaluations for those alternatives and prefer A over B while B should have a better score for that criterion.

Exercise 11
The exhaustivity says that $g_j(a) = g_j(b) \, \forall j \Rightarrow aIb$. Let us take an example where the exhaustivity is not respected: buying a car. Let us consider that the color is the only

criterion and that you prefer blue over all the other colors. The exhaustivity hypothesis would induce that you are indifferent to any blue car, which can hardly be the case since you will probably consider other criteria such as the comfort, the design, the engine, etc.

Exercise 12
See Course about MCDA, slide 79

Exercise 13
See Course about MCDA, slide 78

Exercise 14
This optimisation problem is to minimize the variance while respecting two constraints:

$$\min_X \sigma_p^2 = X\Sigma X'$$

with the constraints:

$$\begin{aligned} \sum_{i=1}^{N} X_i &= 1 \text{ (that we will note } X\mathbf{1} = 1) \\ \mu_p &= X\mu \end{aligned}$$

The two constraints can be rewritten as follows:

$$\begin{aligned} 1 - X\mathbf{1} &= 0 \\ \mu_p - X\mu &= 0 \end{aligned}$$

This problem can be solved by using the Lagrange multipliers:

$$L(X, \lambda_1, \lambda_2) = X\Sigma X' + \lambda_1(\mu_p - X\mu) + \lambda_2(1 - X\mathbf{1}) \tag{17}$$

To find the optimal portfolios, we must solve these equations:

$$\begin{aligned} \nabla_X L &= \Sigma X_p' - \lambda_1 \mu - \lambda_2 \mathbf{1} = 0 && (18) \\ \frac{\partial L}{\partial \lambda_1} &= \mu_p - X_p \mu = 0 && (19) \\ \frac{\partial l}{\partial \lambda_2} &= 1 - X_p \mathbf{1} = 0 && (20) \end{aligned}$$

From (18) we have

$$X_p = \lambda_1(\Sigma^{-1}\mu) + \lambda_2(\Sigma^{-1}\mathbf{1}) \tag{21}$$

Thus we have for (19) and (20)

$$\begin{aligned} (\mu'\Sigma^{-1}\mu)\lambda_1 + (\mu'\Sigma^{-1}\mathbf{1})\lambda_2 &= \mu_p && (22) \\ (\mathbf{1}'\Sigma^{-1}\mu)\lambda_1 + (\mathbf{1}'\Sigma^{-1}\mathbf{1})\lambda_2 &= 1 && (23) \end{aligned}$$

Using that

$$(\mu'\Sigma^{-1}\mathbf{1}) = (\mu'\Sigma^{-1}\mathbf{1})' = \mathbf{1}'(\Sigma^{-1})'\mu = \mathbf{1}'\Sigma^{-1}\mu \tag{24}$$

we can express (22) and (23) as

$$\begin{bmatrix} B & A \\ A & C \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu_p \\ 1 \end{bmatrix} \tag{25}$$

where

$$\begin{bmatrix} B & A \\ A & C \end{bmatrix} = \begin{bmatrix} \mu'\Sigma^{-1}\mu & \mu'\Sigma^{-1}\mathbf{1} \\ \mathbf{1}'\Sigma^{-1}\mu & \mathbf{1}'\Sigma^{-1}\mathbf{1} \end{bmatrix} \tag{26}$$

To ensure there is a solution, the determinant must be non-zero (can be proven as exercise):

$$D = BC - A^2 \neq 0 \tag{27}$$

We can then invert the matrix to obtain:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} A & -C \\ -C & B \end{bmatrix} \begin{bmatrix} \mu_p \\ 1 \end{bmatrix} \tag{28}$$

Thus

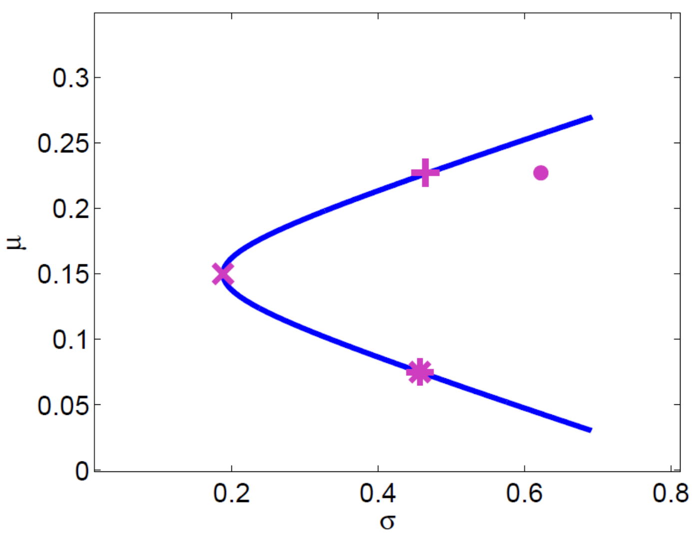
$$\lambda_1 = \frac{C\mu_p - A}{D} \tag{29}$$

$$\lambda_2 = \frac{B - A\mu_p}{D} \tag{30}$$

$$X_p = \frac{C\mu_p - A}{D}\Sigma^{-1} + \frac{B - A\mu_p}{D}\Sigma^{-1}\mathbf{1} \tag{31}$$

$$= \frac{1}{D}(B\Sigma^{-1}\mathbf{1} - A\Sigma^{-1}\mu) + \frac{1}{D}(C\Sigma^{-1}\mu - A\Sigma^{-1}\mathbf{1})\mu_p \tag{32}$$

As an illustration here is the shape of the efficient frontier:



0.3
0.25
0.2
0.15
0.1
0.05
0
μ
0.2
0.4
0.6
0.8
σ