Multicriteria decision aid

Main concepts ?

Dependance to third alternative?

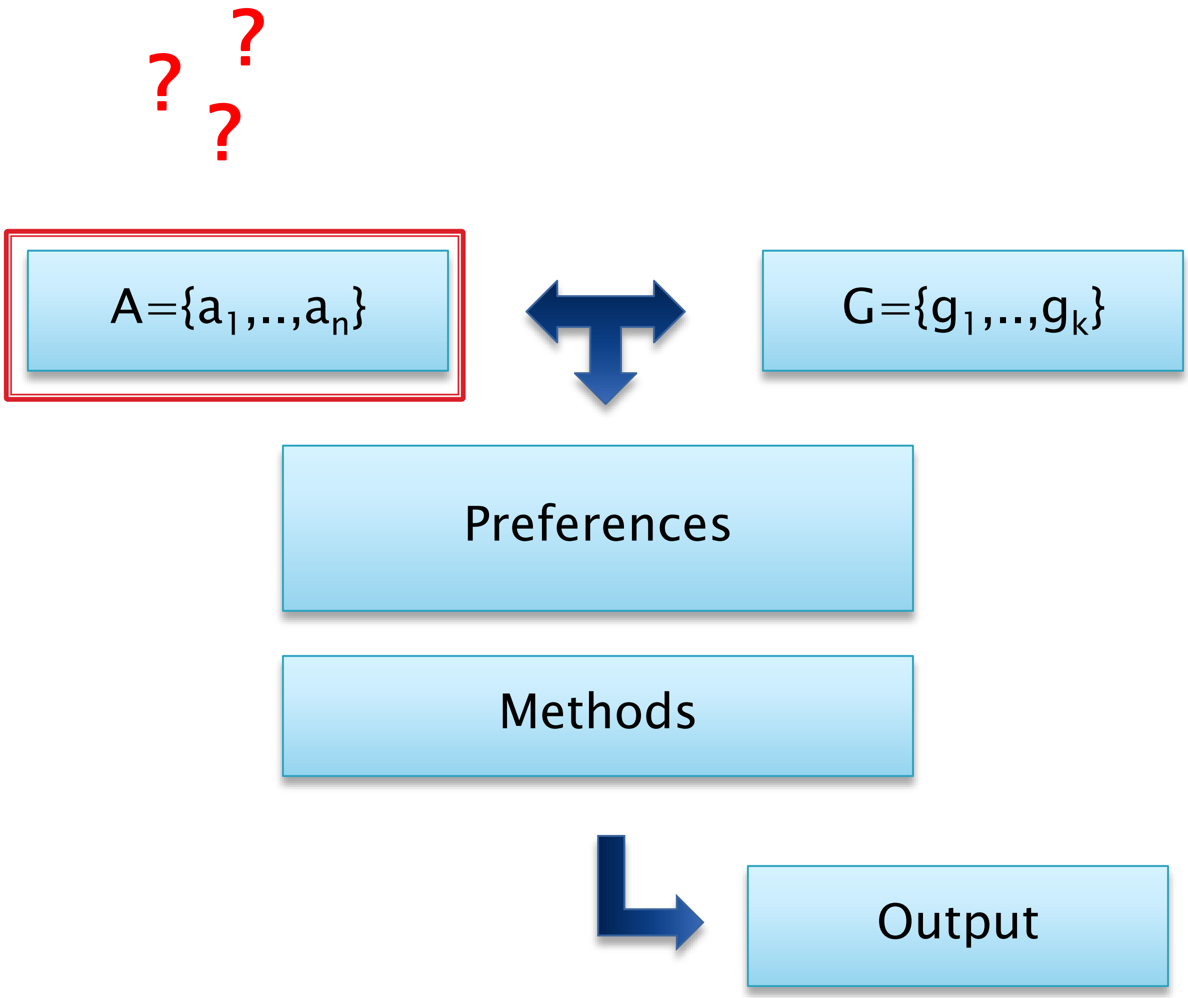
Unanimity ?

Criteria ?

Monotonicity?

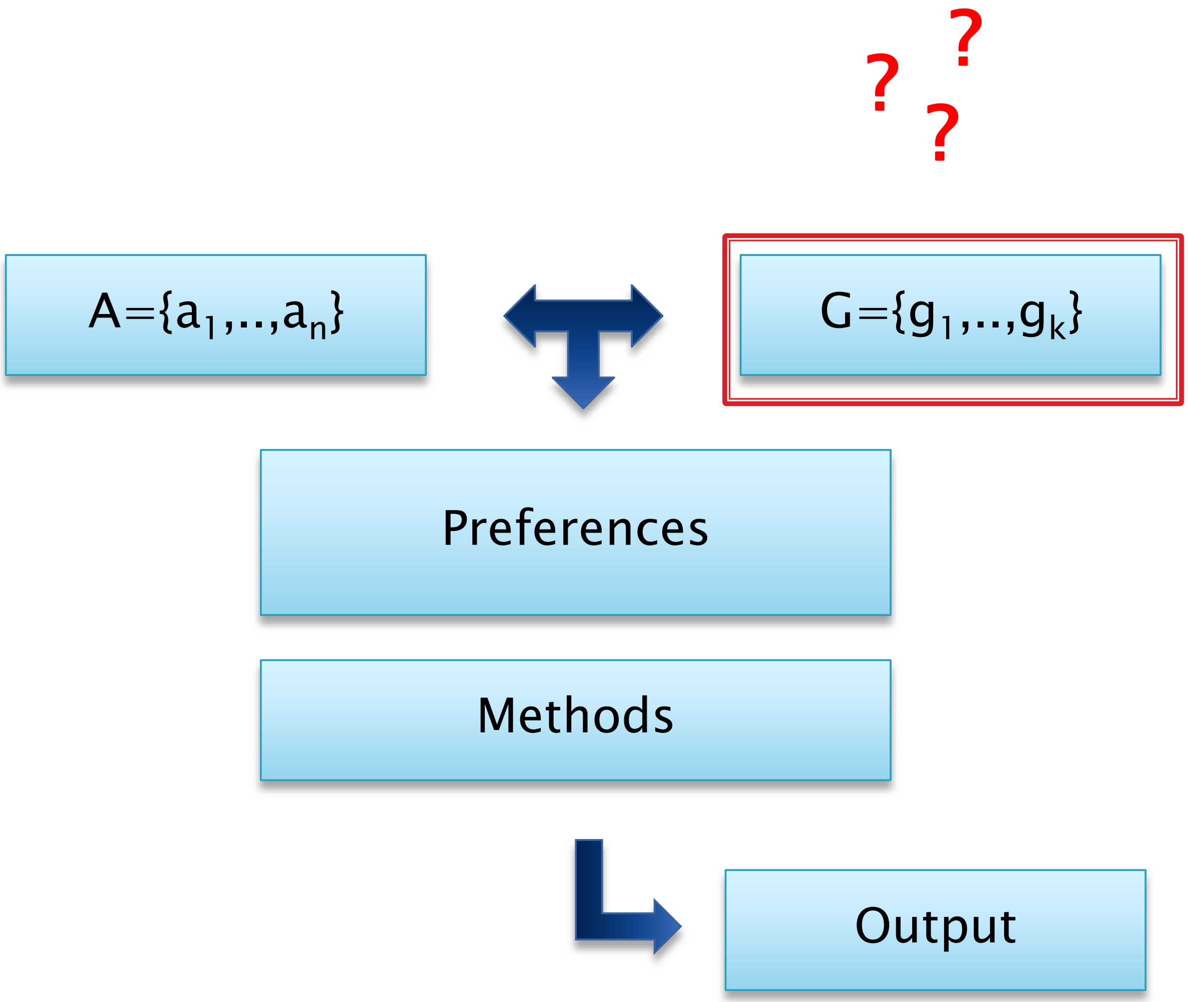
…

Alternatives ?


? ? ?
A={a1,..,an}
G={g1,..,gk}
Preferences
Methods
Output

Alternatives ?

- **A**
- Actions, alternatives, options, items, decisions, …
- Finite ? Countable ? Infinite ?
- Exemple: $A=\{a_1,..,a_n\}$
- Stable / evolutive
  - Ex:
- Defined by extension, by comprehension …
  - Ex:
- Fragmented or globalized
  - Ex:


A={a1,..,an}
G={g1,..,gk}
???
Preferences
Methods
Output

Criteria

- **Definition**: a criterion is a mapping of A into a totally ordered set

$$g_i: A \rightarrow E_i$$

- W.l.g. criteria have to be maximized
- Scales ?
  - Nominal
  - Ordinal $<,=,>$
  - Interval $<,=,>,+,-$
  - Ratio $<,=,>,+,-,/,*$

Dominance

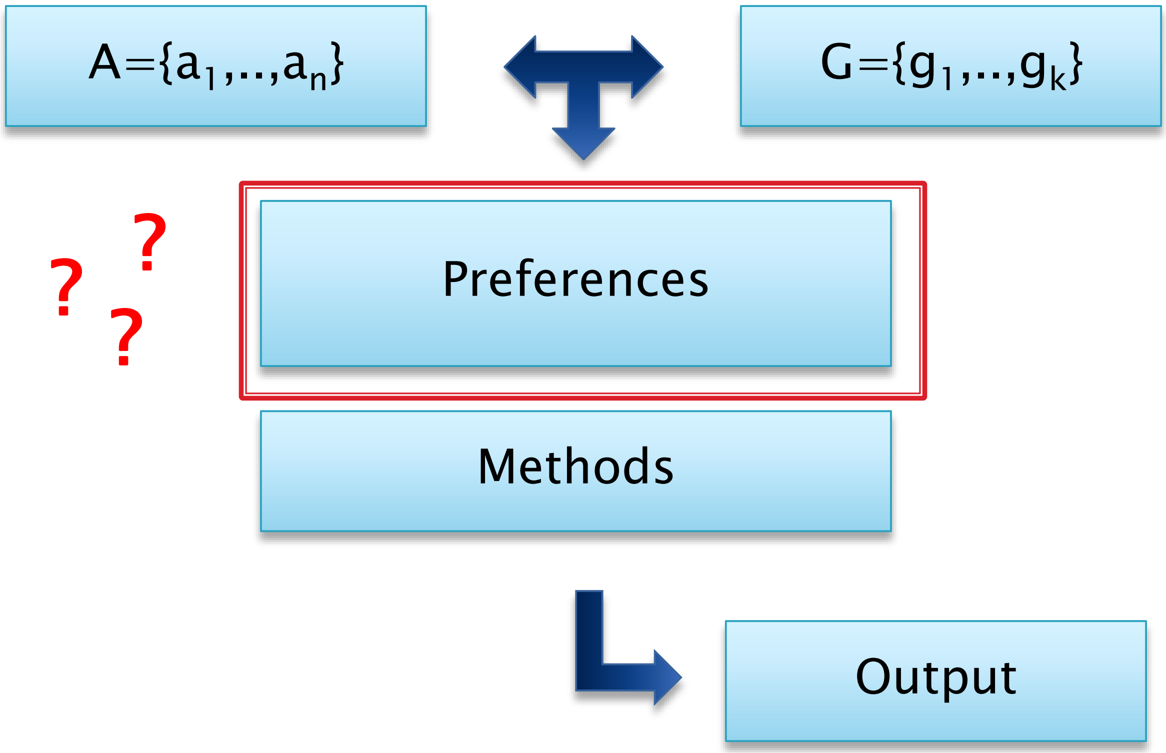
- Unanimity principle
- **<u>Definition</u>**: a is said to dominate b iff $f_i(a) \geq f_i(b)$ and $\exists j \mid f_j(a) \geq f_j(b)$
- A = {efficient solutions} ∪ {dominated solutions}
- PO(A) = Pareto optimal set = {efficient solutions}
- Main problem: #PO(A) ≈ #A
- The identification of PO(A) is often a problem itself …

What's next ?

- Once the PO set has been identified, how can we select the best choice;
- There is no unique objective choice !
  - There are plenty of different cars in the streets !

	Price	Power	Cons.	Habitability	Comfort
Average A	18000	75	8,0	3	3
Sport	18500	**110**	9,0	1	2
Average B	17500	85	**7,0**	4	3
Luxury 1	24000	90	8,5	4	**5**
Economic	**12500**	50	7,5	2	1
Luxury 2	22500	85	9,0	**5**	4

- Central role of the **decision maker**
- **Binary preference** ↔ valued preferences



A={a1,..,an}
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Preference structure

- Main assumption: given two alternatives a and b, the decision maker is able to express one of the following statements:
  - a P b (or b P a): preference
  - a I b: indifference
  - a J b: incomparability
- P is **asymmetric**
- I is **reflexive** and **symmetric**
- J is **irreflexive** and **symmetric**
- What about **transitivity** ? Is it a natural property ?

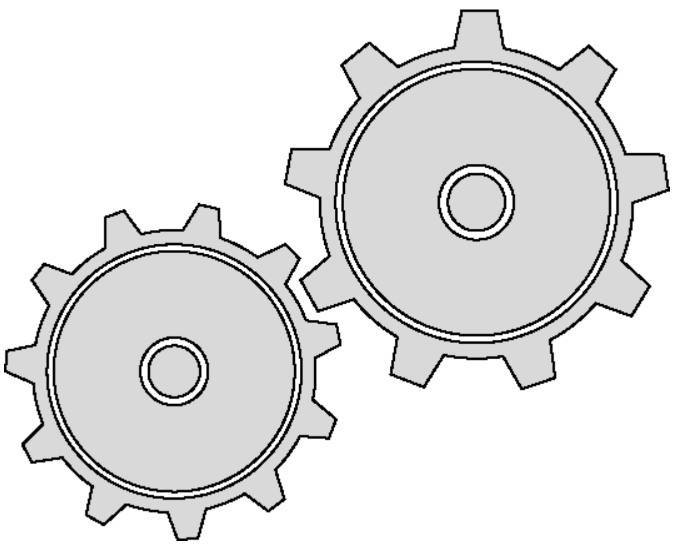
Famous example of Luce (1956)

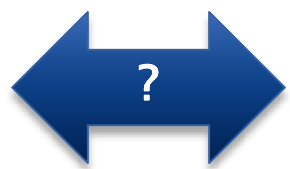
C_0 C_1 C_2 C_3 … C_{401}

The main problem

MCDA model

Decision maker (P,I,J)





How to select the criteria ?

- Clear link between criteria and preferences
- Consistent family of criteria

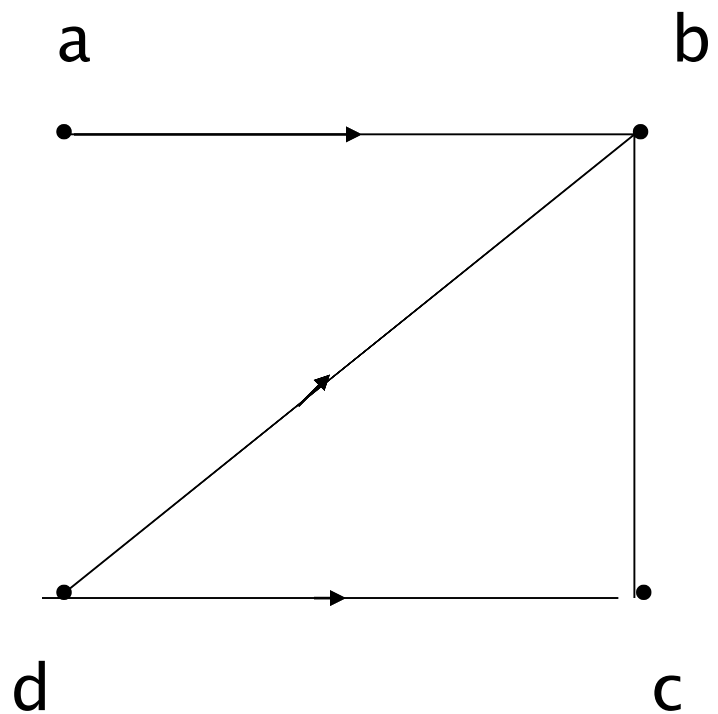
  - Exhaustivity

    $$\text{If } f_i(a)=f_i(b)\ \forall\ i=1,..,q \Rightarrow a\ I\ b$$

  - Cohesion

  - Non redundancy

Graph modelling



a P b

a J c

a J d

b I c

d P b

d P c

Main question:

How can we represent the preferences of a decision maker ?

Traditional structure

- Optimisation of function g defined on A

$$\forall a,b \in A : \begin{cases} aPb & \Leftrightarrow \quad g(a) > g(b) \\ aIb & \Leftrightarrow \quad g(a) = g(b) \end{cases}$$

- Consequences :

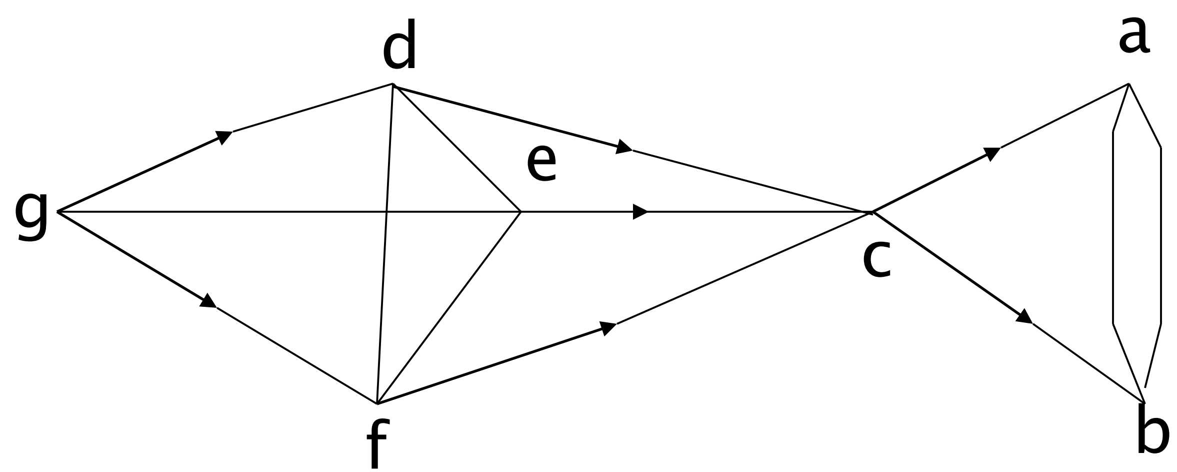
R is empty
P is transitive
I is transitive

- Total pre-order
- Total order (if no ex aequo)

Example

The decision maker prefers an action to another if the « gain » is higher

a	**b**	**c**	**d**	**e**	**f**	**g**
100	100	120	130	130	130	131



Indifference threshold

- Indifference is not transitive (Luce, 1956)
- Indifference threshold:

$$\forall a,b \in A: \begin{cases} aPb & \Leftrightarrow & g(a) > g(b) + q \\ aIb & \Leftrightarrow & \left|g(a) - g(b)\right| \leq q \end{cases}$$

- Semi-order : P is transitive, not I.

Properties (exercice)

- aPb, bPc, aId $\Rightarrow$ dPc
- aPb, bIc, cPd $\Rightarrow$ aPd

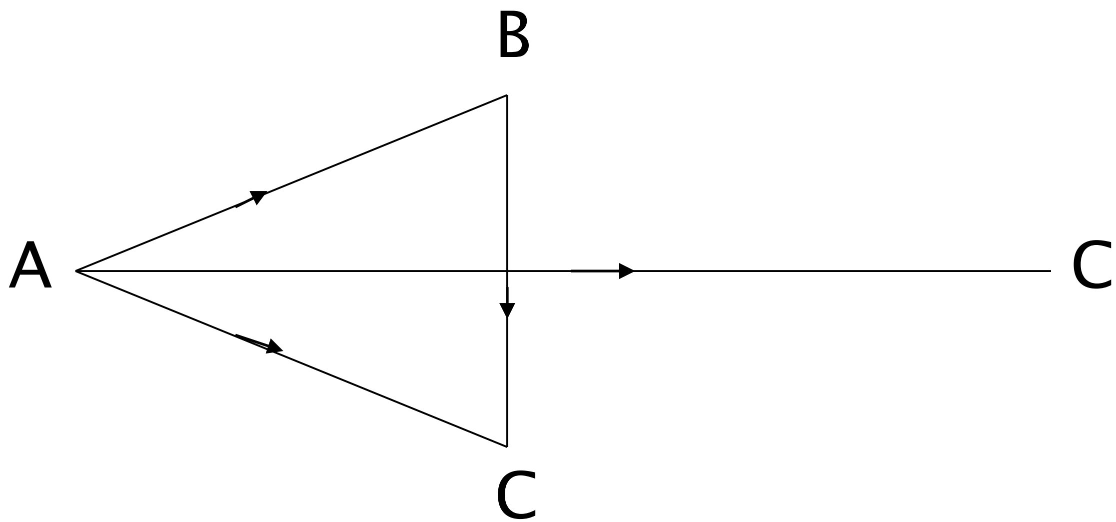
Other preference structures

- Variable indifference thresholds
  $\Rightarrow$ interval order.
- Indifference + preference thresholds
  $\Rightarrow$ Pseudo-order
- Including incomparability
  $\Rightarrow$ Partial orders
- Valued preferences

Including incomparability

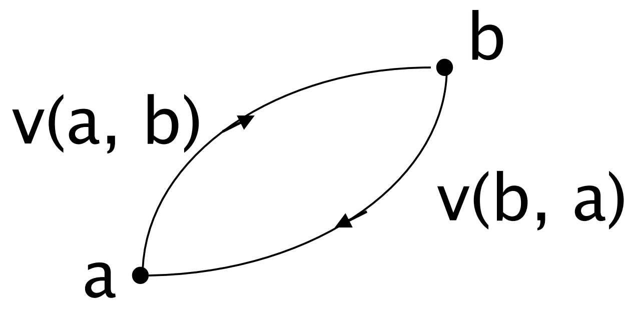
The decision maker prefers an investment to another if the estimations of all the experts are compatible with this judgement

	A	B	C	D
1rst expert	10	8	7	6
2nd expert	9	7	5	6
3rd expert	12	8	9	4



Partial Order : P is transitive but J is not empty

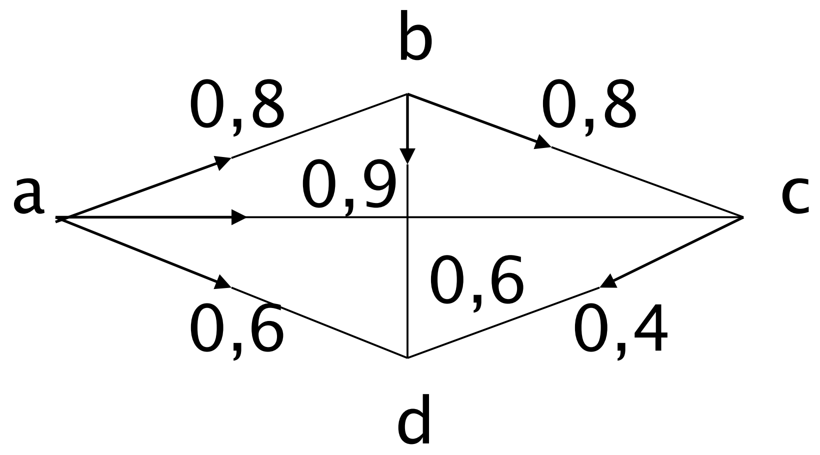
Valued preferences



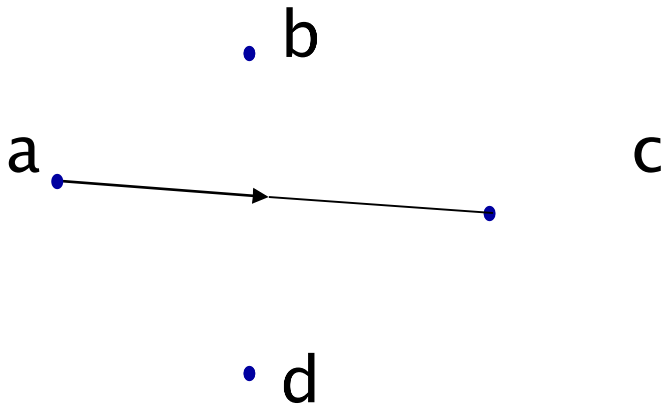
– Preference intensity

– Proportion of votes

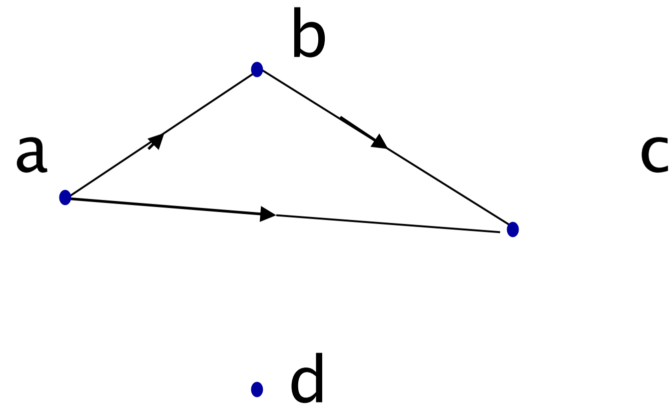
Example



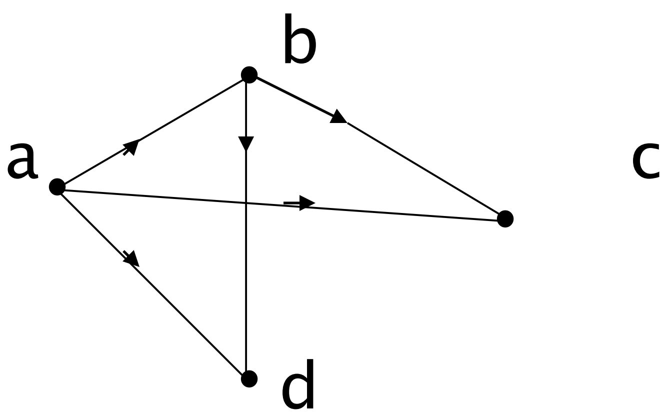
Cutting level 0,9



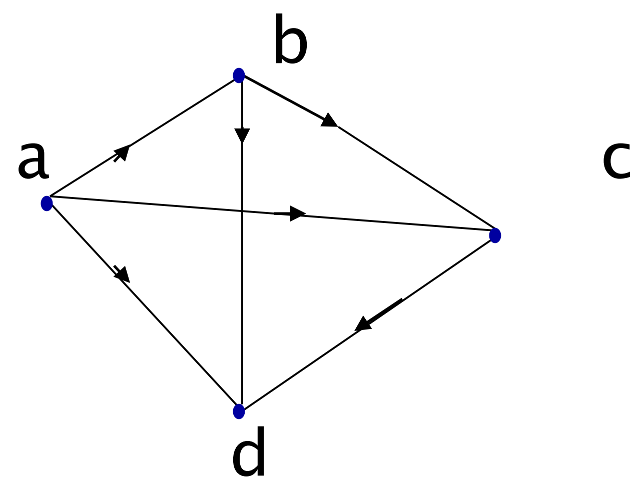
Cutting level 0,8



Cutting level 0,6



Cutting level 0,4



Preferential independence

$J \subset G$ is preferentially independent within G if $\forall$ a, b, c, d $\in$ A such that

$$\begin{cases} g_j(a) = g_j(b), \forall j \notin j \\ g_j(c) = g_j(d), \forall j \notin j \\ g_j(a) = g_j(c), \forall j \in j \\ g_j(b) = g_j(d), \forall j \in j \end{cases}$$

We have a P b $\Leftrightarrow$ c P d

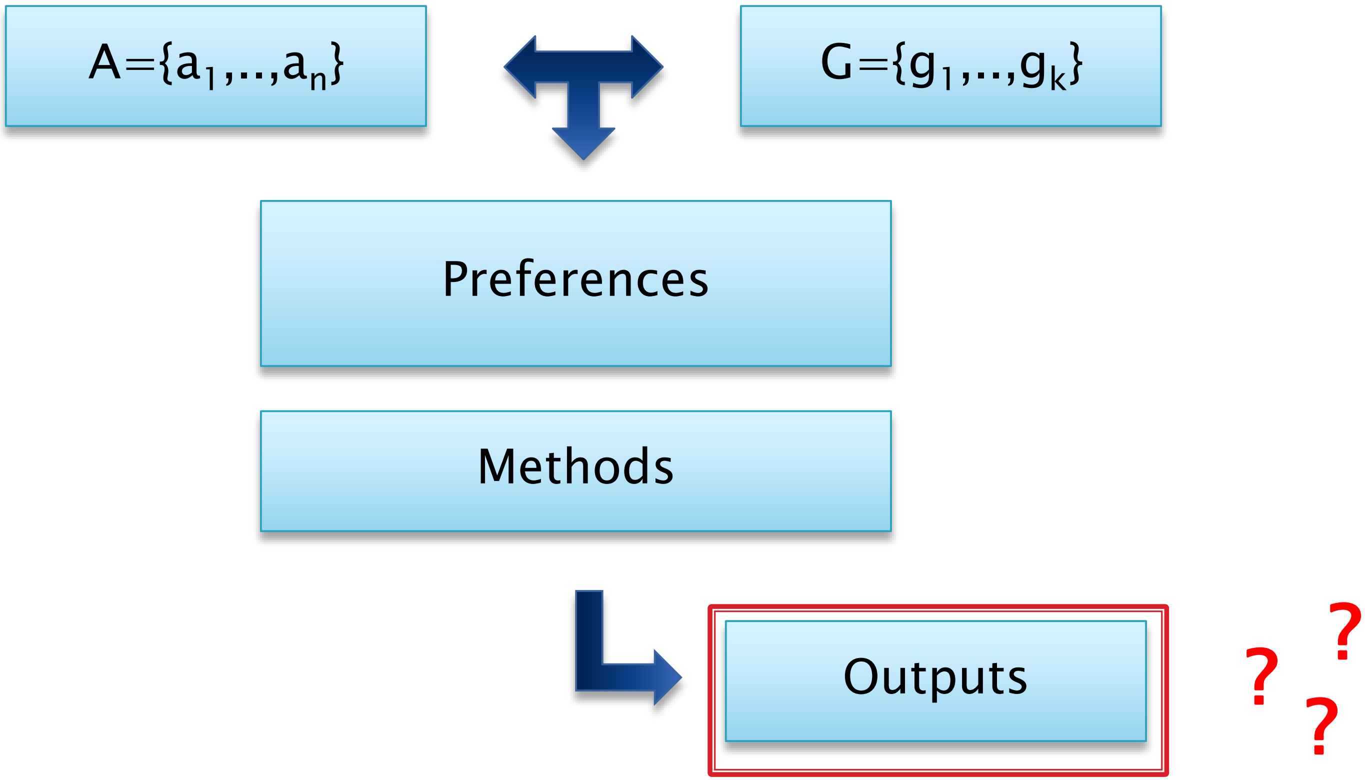
Exemple

	a	b	c	d
g_1	45	50	45	50
g_2	70	70	90	90
g_3	100	80	100	80

The criteria $\{g_1, g_3\}$ constitute a sub-family of preferentially independent criteria if a P b $\Leftrightarrow$ c P d

Counter-example

	Dish	Wine
a	Fish	White
b	Meat	White
c	Fish	Red
d	Meat	Red


A={a1,..,an}
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? ? ?

« Problematics »

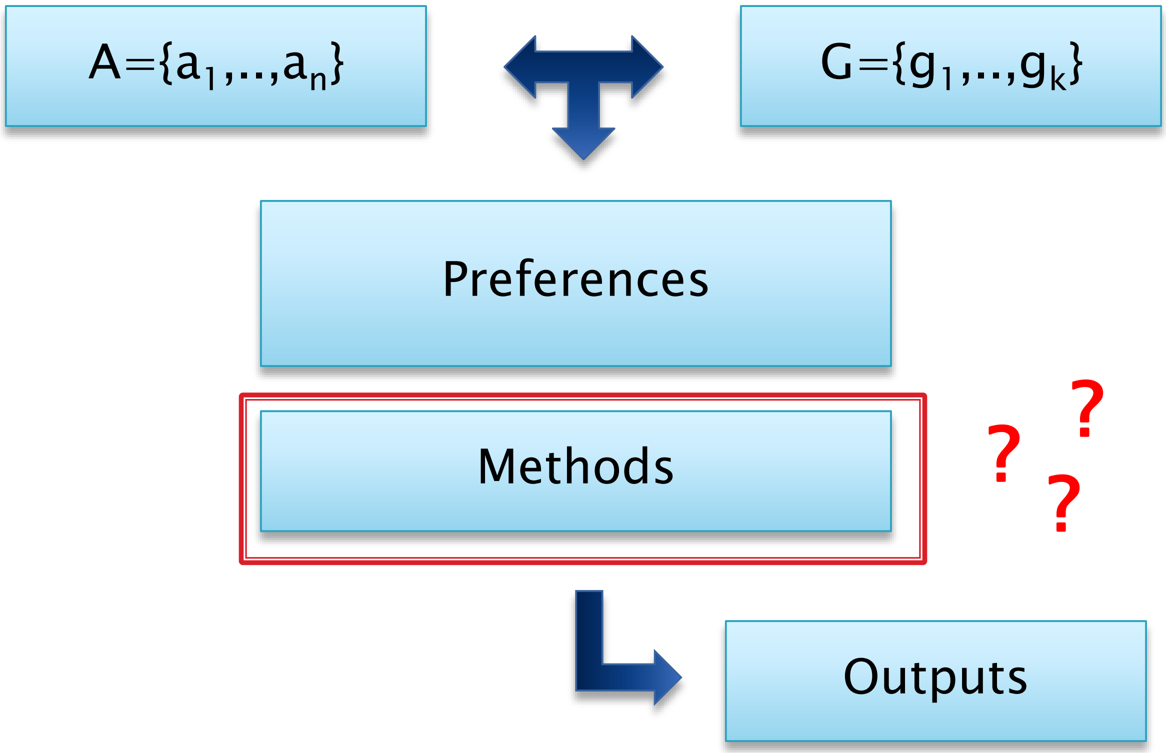
	g_1	g_2	g_3	...
a	$g_1(a)$	$g_2(a)$	$g_3(a)$	...
b	$g_1(b)$	$g_2(b)$	$g_3(b)$	...
c	...			
...	...			

α – <u>Choice</u>: to determine a subset of « good » actions

β – <u>Sorting</u>: to sort the actions in pre-defined categories

γ – <u>Ranking</u>: to rank the actions from the best to the worst

δ – <u>description</u> : to describe the actions and their consequences



$A=\{a_1,..,a_n\}$
$G=\{g_1,..,g_k\}$
Preferences
Methods
? ? ?
Outputs

Three main families of methods

- Multi-Attribute Utility Theory (MAUT)
- Interactive methods
- Outrankings methods
  - ELECTRE methods
  - PROMETHEE methods

Multi-Attribute Utility Theory (MAUT)

- A unique criterion.

$$U(a) = U\big(g_1(a), g_2(a), \ldots, g_k(a)\big)$$

- Existence ?
- Construction ?
- Form ?

➔ additive ?

$$U(a) = \sum_{j=1}^{k} U_j\big(g_j(a)\big)$$

Example – additive model

	a	b	c	d	
Price	300.000	350.000	400.000	450.000	
Comfort	medium	good	good	very good	
U_1	8,5	8	6	5	$k_1 = 7$
U_2	4	7	7	10	$k_2 = 3$

Example

$$U(a) = 7 \times 8{,}5 + 3 \times 4 = 71{,}5$$

$$U(b) = 7 \times 8 + 3 \times 7 = 77$$

$$U(c) = 7 \times 6 + 3 \times 7 = 63$$

$$U(d) = 7 \times 5 + 3 \times 10 = 65$$

Marginal utilities

Let X_j be the set of possible values for criterion g_j

Let x_j and y_j the worst and the best values

Method 1

Determine z_j « middle » between x_j and y_j

v_j « middle » between x_j and z_j

w_j « middle » between z_j and y_j

…

$\Rightarrow$ $U_j(z_j) = 1/2\ [U_j(x_j) + U_j(y_j)]$

$U_j(v_j) = 1/2\ [U_j(x_j) + U_j(z_j)]$

Marginal utilities

Method 2

Determine z_j indifferent to

x_j with prob. 1/2

y_j with prob. 1/2

$$\Rightarrow \quad U_j(z_j) = 1/2\,[U_j(x_j) + U_j(y_j)]$$

Determine v_j indifferent to

x_j with prob. 1/2

z_j with prob. 1/2

$$\Rightarrow \quad U_j(v_j) = 1/2\,[U_j(x_j) + U_j(z_j)]$$

Marginal utilities

Method 3

Determine $z_j(p)$ indifferent to
- x_j with prob. p
- y_j with prob. $(1-p)$

Let p vary.

$$\Rightarrow \quad U_j(z_j(p)) = pU_j(x_j) + (1-p)\, U_j(y_j)$$

An original idea: AHP

- Verbal scale for pair-wise comparison:

Verbal	Equal	Moderate	Strong	Very Strong	Extreme
Numeric	1	3	5	7	9

Criteria A	Criteria B	More Important	Intensity
Cost	Safety	A	3
Cost	Style	A	7
Cost	Capacity	A	3
Safety	Style	A	9
Safety	Capacity	A	1
Style	Capacity	B	7

Source wikipédia

	cost	safety	capacity	style
cost	1	3	3	7
safety	0,333333	1	1	9
capacity	0,333333	1	1	0,142857
style	0,142857	0,111111	7	1

An original idea: AHP

- Compute the eigen vector
- Consistency ?

$$w_{ji} = w_{ij}^{-1}$$

$$w_{ij} = w_{ik} . w_{kj}$$

	cost	safety	capacity	style
cost	1	3	3	7
safety	0,333333	1	1	9
capacity	0,333333	1	1	0,142857
style	0,142857	0,111111	7	1

0,7787	0,5457	0,1655	0,2614
0,444641	**0,311597**	**0,094501**	**0,149261**

0,522653	0,366266	0,111081

	cost	safety	capacity	style
cost	1	3	3	7
safety	0,333333	1	1	9
capacity	0,333333	1	1	0,142857
style	0,142857	0,111111	7	1

0,9045	0,3015	0,3015
0,6	0,2	0,2

An original idea : AHP

	cost	safety	capacity	style
cost	1	3	3	7
safety	0,333333	1	1	9
capacity	0,333333	1	1	0,142857
style	0,142857	0,111111	7	1

0,7787	0,5457	0,1655	0,2614
0,444641	**0,311597**	**0,094501**	**0,149261**

	cost	safety	capacity	style
cost	1	3	3	1
safety	0,333333	1	1	9
capacity	0,333333	1	1	0,142857
style	1	0,111111	7	1

0,5784	0,7048	0,1816	0,3683
0,315531	0,384485	0,099067	0,200916

Three main families of methods

- Multi-Attribute Utility Theory (MAUT)
- Interactive methods
- Outrankings methods
  - ELECTRE methods
  - PROMETHEE methods

Pairwise comparisons

Electre I

Condorcet principle

p_j = weight of criterion j ; j = 1,…, n.

Concordance index

$$c(a,b) = \frac{1}{P} \sum_{j \mid g_j(a) \geq g_j(b)} p_j$$

Discordance index

$$d(a,b) = \begin{cases} 0 \text{ si } g_j(a) \geq g_j(b),\ \forall j \\ \frac{1}{\delta} \max_j [g_j(b) - g_j(a)] \text{ sinon} \end{cases}$$

$$\text{où } \delta = \max_{j,\,c,\,d} [g_j(c) - g_j(d)]$$

Electre I

- Another option…

D_j = {(x_j, y_j) : g_j(a) = x et g_j(b) = y ⇒ a does not outrank b}

Outraking relation

a S b iff $\begin{cases} c(a, b) \geq \hat{c}, \\ d(a, b) \leq \hat{d}; \end{cases}$

or

a S b iff $\begin{cases} c(a, b) \geq \hat{c} \\ (g_j(a), g_j(b)) \notin D_j, \forall j \end{cases}$

Electre I – Kernel

The kernet is a subset N such that

$$\begin{cases} \forall b \notin N : \exists a \in N : a\,S\,b \\ \forall a, b \in N : a \not S\, b. \end{cases}$$

<u>N.B.</u> If S is acyclic $\Rightarrow$ N exists and is unique

ELECTRE I : numerical example

	Price	Comfort	Speed	Esthetic
1	300	excellent	fast	good
2	250	excellent	medium	good
3	250	medium	fast	good
4	200	medium	fast	medium
5	200	medium	medium	good
7	100	poor	medium	medium
Poids	5	4	3	3

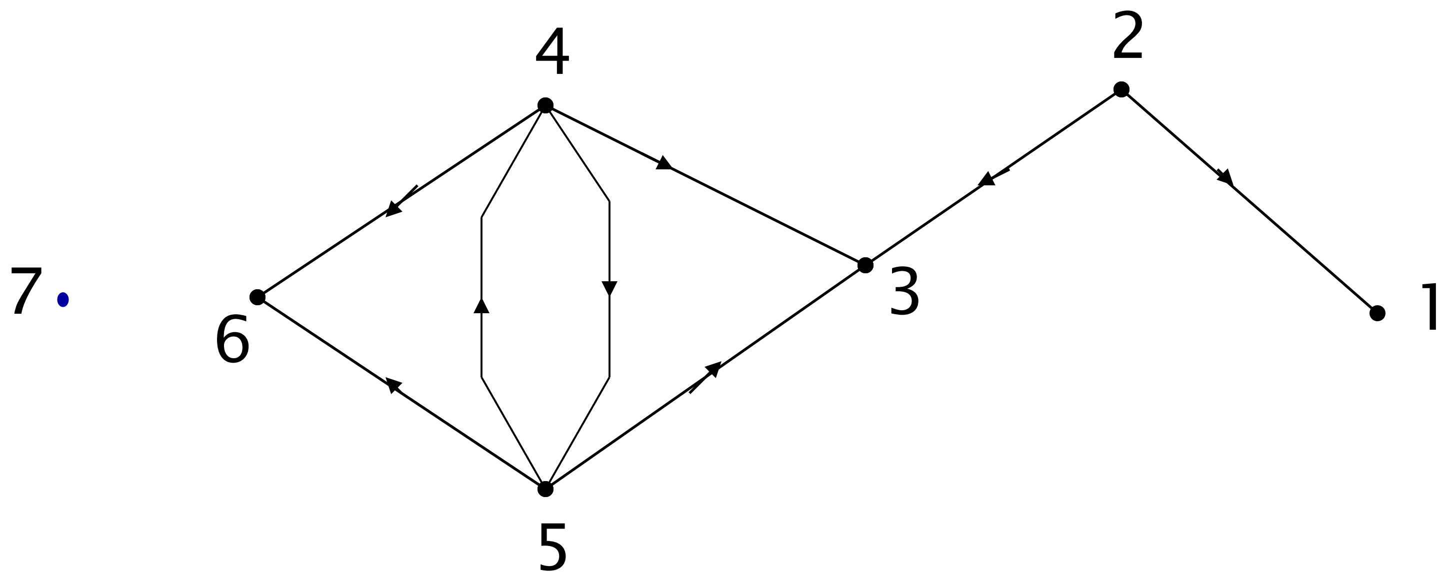
Concordance index

	1	2	3	4	5	6	7
1	–	10	10	10	10	10	10
2	12	–	12	7	10	7	10
3	11	11	–	10	10	10	10
4	8	8	12	–	12	12	10
5	8	11	12	12	–	12	10
6	11	11	11	11	11	–	10
7	5	8	5	8	8	9	–

Discordance

	Price	Price	Comfort
a	250	300	poor
b	100	100	Excellent

Outranking relation for a concordance threshold equal to 12



Kernel: 2, {4, 5}, 7

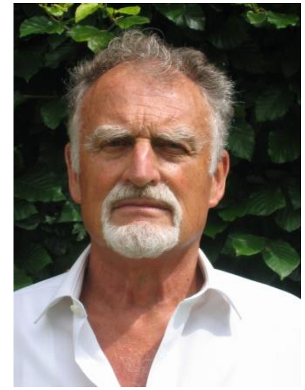
PROMETHEE & GAIA

Outline

- **Introduction**
- A **pedagogical** example
- PROMETHEE I & II **rankings**
- **Properties**
- A few words about **rank reversal**
- **GAIA**
- Software demonstration
- Conclusion

Historical background

Prof. Jean-Pierre Brans
(VUB, Solvay School)



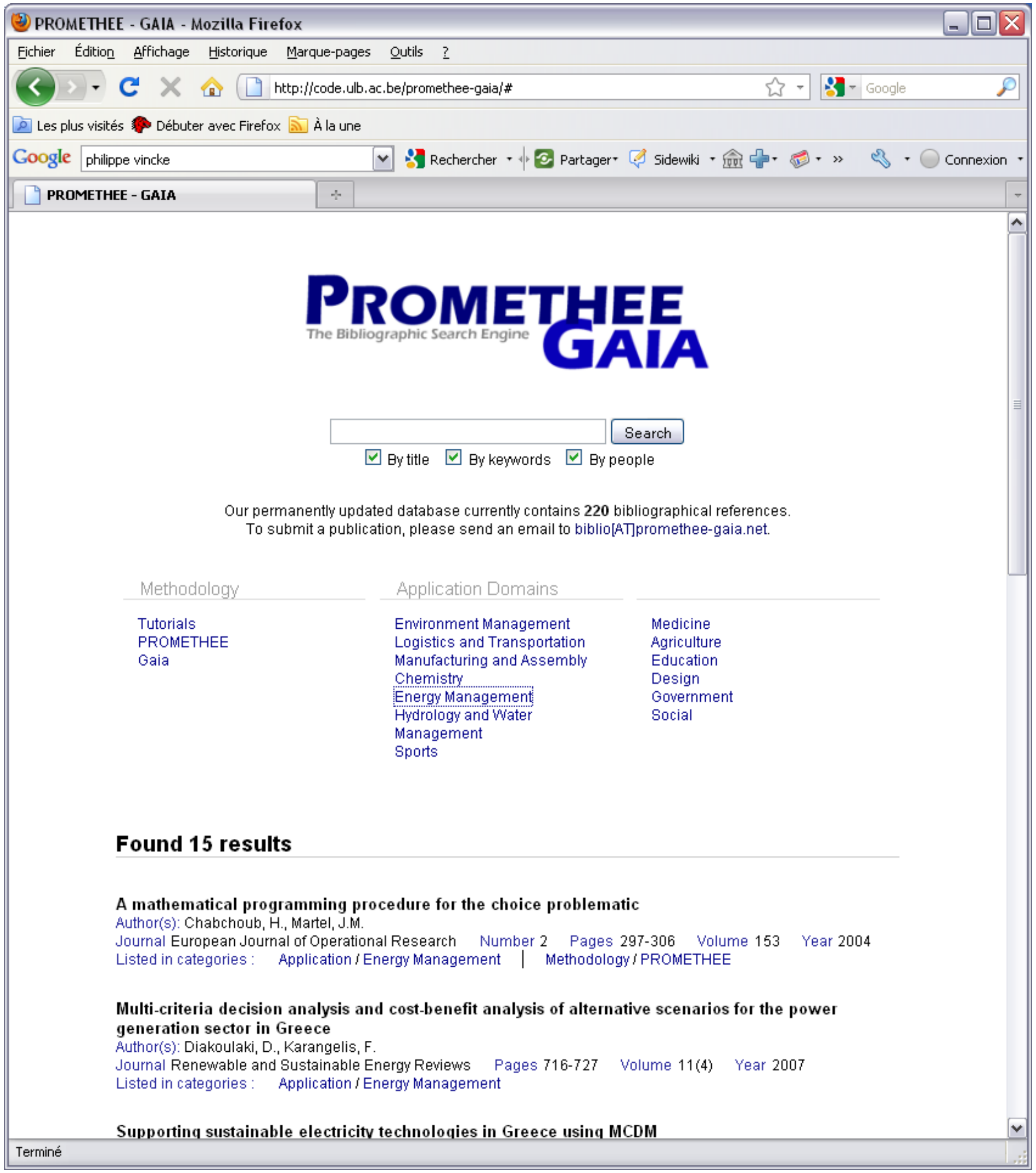
Prof. Philippe Vincke
(ULB, Engineering Faculty)

Prof. Bertrand Mareschal
(ULB, Solvay Brussels School of Economics and Management)

Applications

Behzadian, M.; Kazemzadeh, R.B.; Albadvi, A.; Aghdasi, M. (2010) « *PROMETHEE: A comprehensive literature review on methodologies and applications* », *EJOR, Vol.200(1),198-215*

- > 200 papers published in > 100 journals
- **Topics**: Environmental management, hydrology and water management, finance, chemistry, logistics and transportation, energy management, health care, manufactoring and assembly, sports,…



http://code.ulb.ac.be/promethee-gaia

Let us agree on a few points

- Multicriteria decision problems are **ill-defined** (no optimal solutions);
- Decision **aid** versus decision **making**;
- « *The Truth is Out There* » (X-Files);
- « *The purpose of models is not to fit the data but to sharpen the questions* », Samuel Karlin

Let us start with a educational example !

An educational example

- A plant location problem
  - 6 possible locations
  - 6 criteria

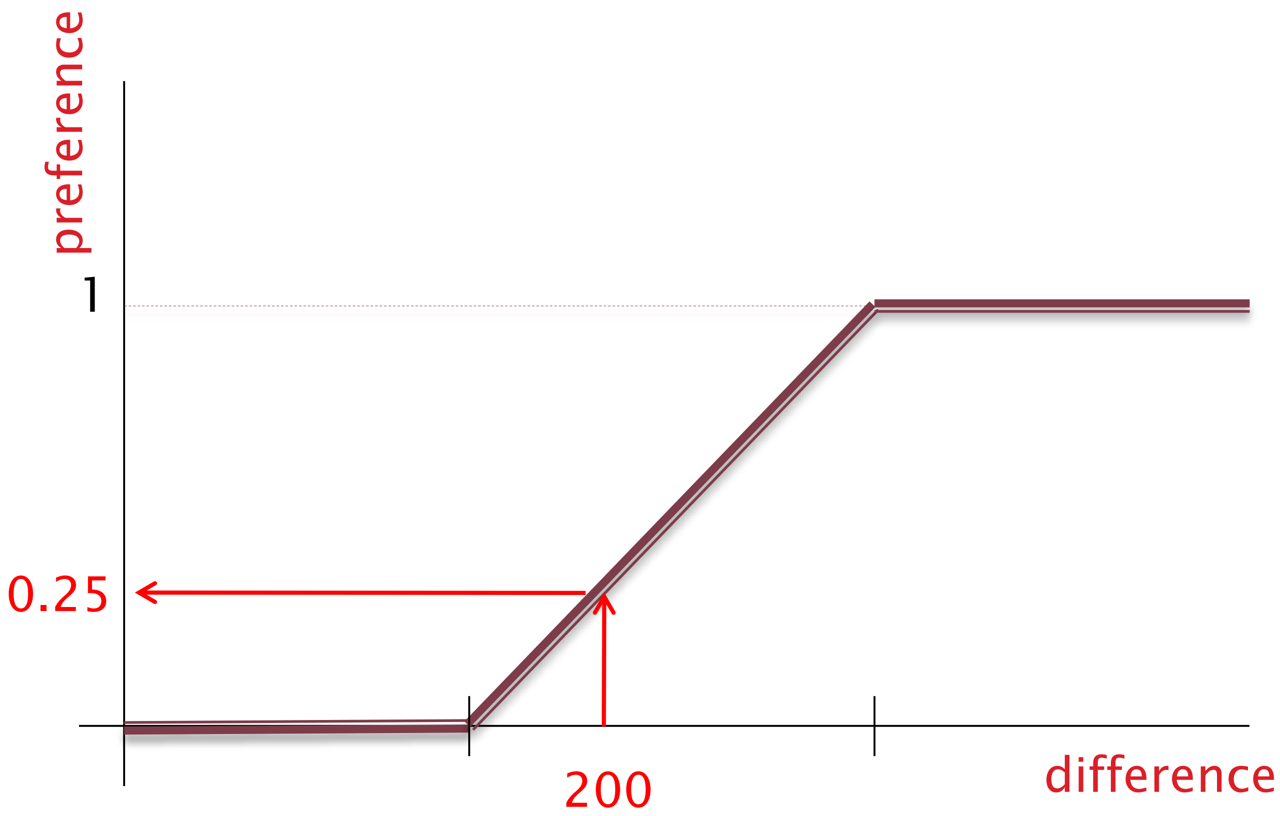
Units	10MW	10^6\$	10^6\$			
	Engineers	Power	Cost	Maintenance	Village	Security
Italy	75	90	600	5,4	**8**	5
Belgium	65	**58**	**200**	**9,7**	**1**	**1**
Germany	83	60	400	7,2	4	7
Sweden	**40**	80	**1.000**	7,5	7	**10**
Austria	52	72	600	**2**	3	8
France	**94**	**96**	700	3,6	5	6

Main principle: pair-wise comparisons

	Engineers	Power	Cost	Maintenance	Village	Security
Italy	75	90	600	5,4	8	5
Belgium	65	58	200	9,7	1	1
Germany	83	60	**400**	7,2	4	7
Sweden	40	80	1.000	7,5	7	10
Austria	52	72	**600**	2	3	8
France	94	96	700	3,6	5	6

- Concerning the cost, **Germany is better than Austria** !
- How can we **quantify** this advantage ? 200 ?
- What does it **mean** ?

Unicriterion preference function



Step 1: compute unicriterion preference degree for every pair of alternatives

			0.25			
			-200			
Germany	83	60	400	7,2	4	7
	Engineers	Power	Cost	Maintenance	Village	Security
Austria	52	72	600	2	3	8
	-31	12		-5.2	-1	1
	1	0.75		1	0.3	0.63

Step 2: compute global preference degree for every pair of alternatives

			0.25			
Germany	83	60	400	7,2	4	7
	Engineers	Power	Cost	Maintenance	Village	Security
? **Weights**	**0.1**	**0.2**	**0.2**	**0.1**	**0.15**	**0.15**
Austria	52	72	600	**2**	3	8
	1	**0.75**		**1**	**0.3**	**0.63**

Π(Austria, Germany)$= 1*0.1 + 0.75*0.2 + 1*0.1 + 0.3*0.15 + 0.63*0.15$
$= \mathbf{0.489}$

Π(Germany,Austria)$= 0.25*0.4 = \mathbf{0.05}$

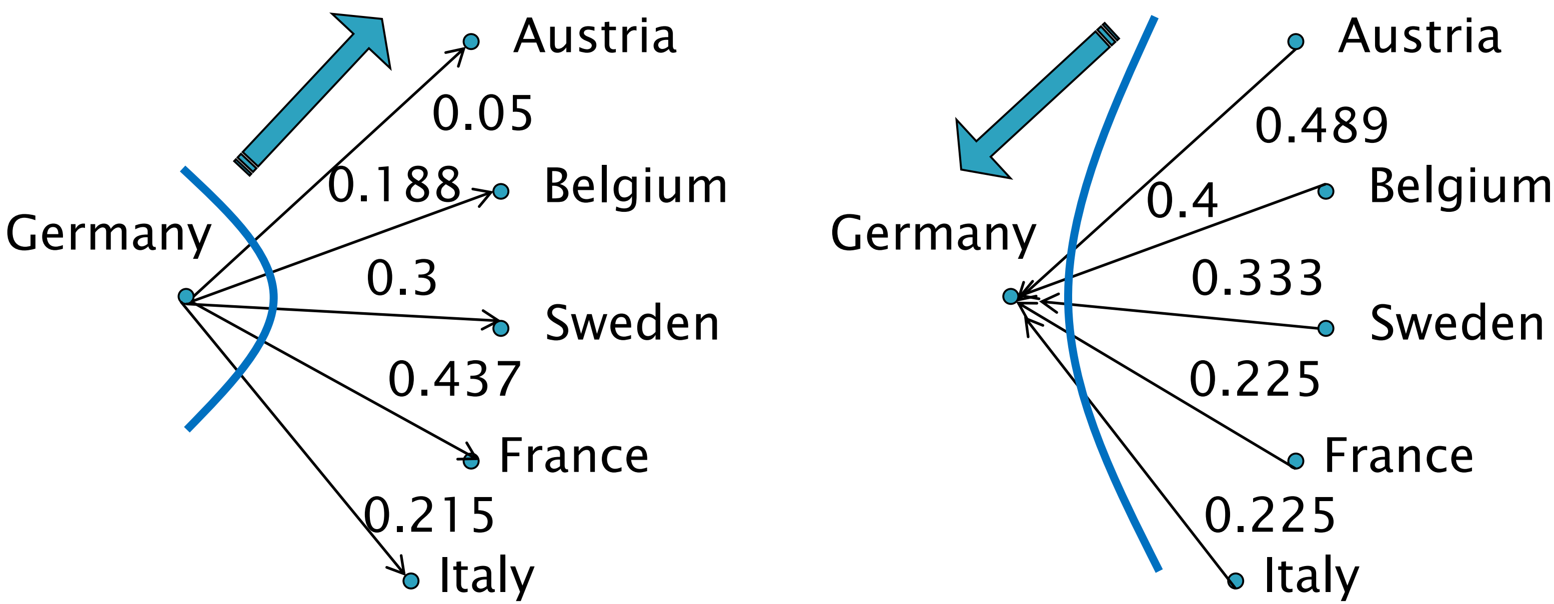
Preference matrix

Preference Degrees

	Italy	Belgium	Germany	Sweden	Austria	France
Italy	0,000	0,280	0,225	0,242	0,090	0,217
Belgium	0,267	0,000	0,400	0,300	0,267	0,500
Germany	0,215	0,188	0,000	0,300	0,050	0,437
Sweden	0,429	0,545	[illegible]	0,000	0,255	0,255
Austria	0,458	0,545	0,489	0,342	0,000	0,457
France	0,259	0,379	0,225	0,388	0,120	0,000

- How can we **exploit** this matrix ?
- … in order to obtain a **ranking** (complete or partial) ?

Step 3: compute positive, negative and net flow scores



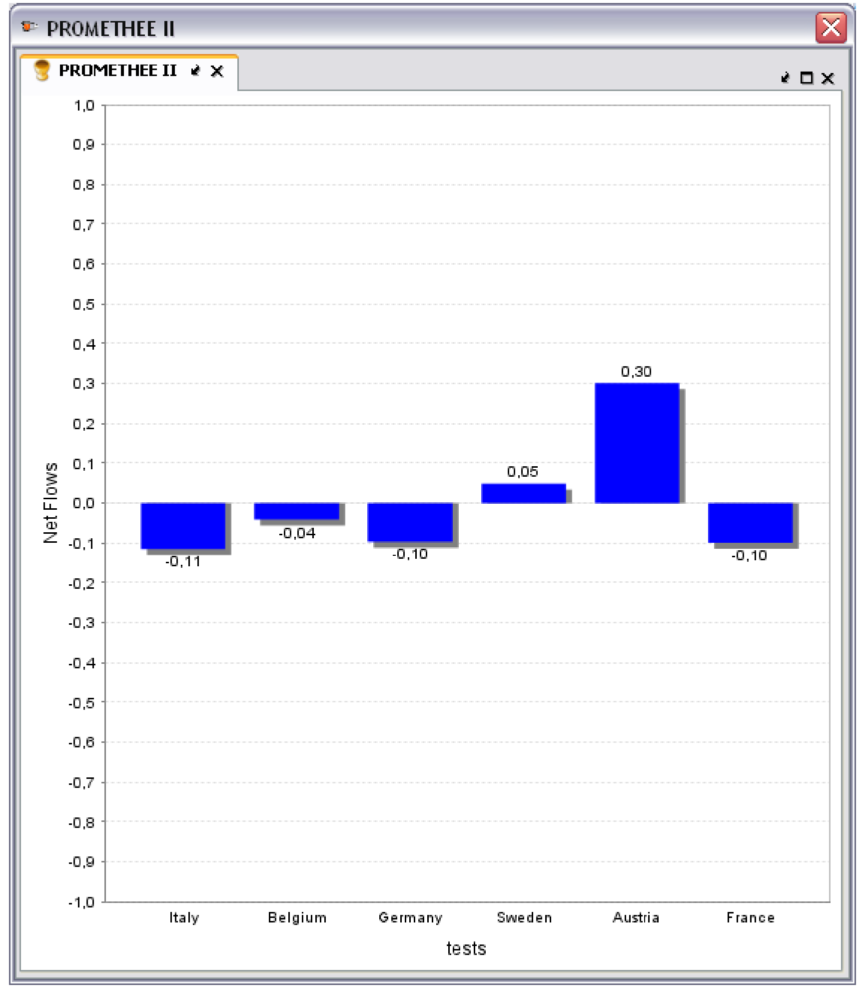
$$\Phi^{+}(Germany) = 0.238 \qquad \Phi^{-}(Germany) = 0.334$$

$$\Phi(Germany) = \Phi^{+}(Germany) - \Phi^{-}(Germany) = -0.1$$

PROMETHEE II

Flows

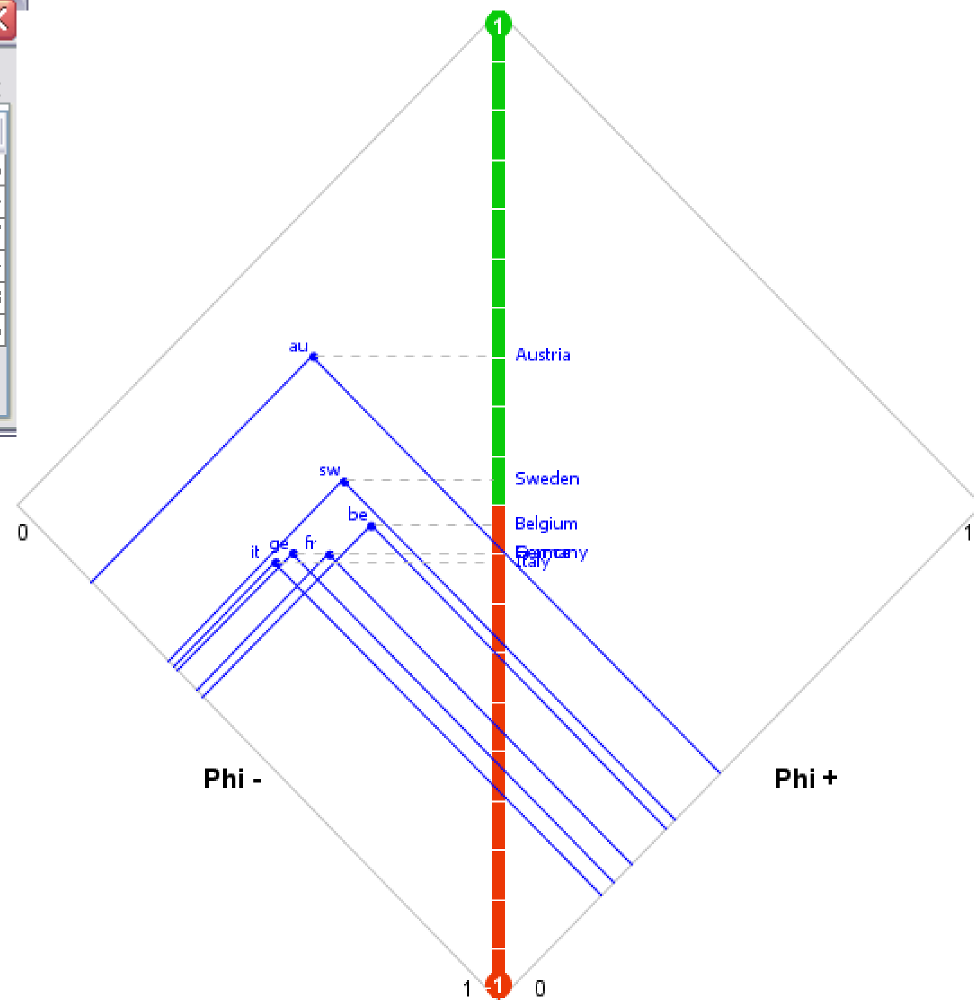
Alternative	Rank	Net Flow	Positive Flow	Negative Flow
Austria	1	0,302	0,458	0,156
Sweden	2	0,049	0,363	0,314
Belgium	3	-0,041	0,347	0,387
Germany	4	-0,096	0,238	0,334
France	5	-0,099	0,274	0,373
Italy	6	-0,115	0,211	0,326



PROMETHEE I

Flows

Alternative	Rank	Net Flow	Positive Flow	Negative Flow
Austria	1	0,302	0,458	0,156
Sweden	2	0,049	0,363	0,314
Belgium	3	-0,041	0,347	0,387
Germany	4	-0,096	0,238	0,334
France	5	-0,099	0,274	0,373
Italy	6	-0,115	0,211	0,326



Formalization

PROMETHEE

Preference **R**anking **O**rganisation
METHod for **E**nrichment **E**valuations

Formalization

- A **finite** set of alternatives

$$A=\{a_1,a_2,..,a_n\}$$

- A set of (**quantitative** or **qualitative**) criteria

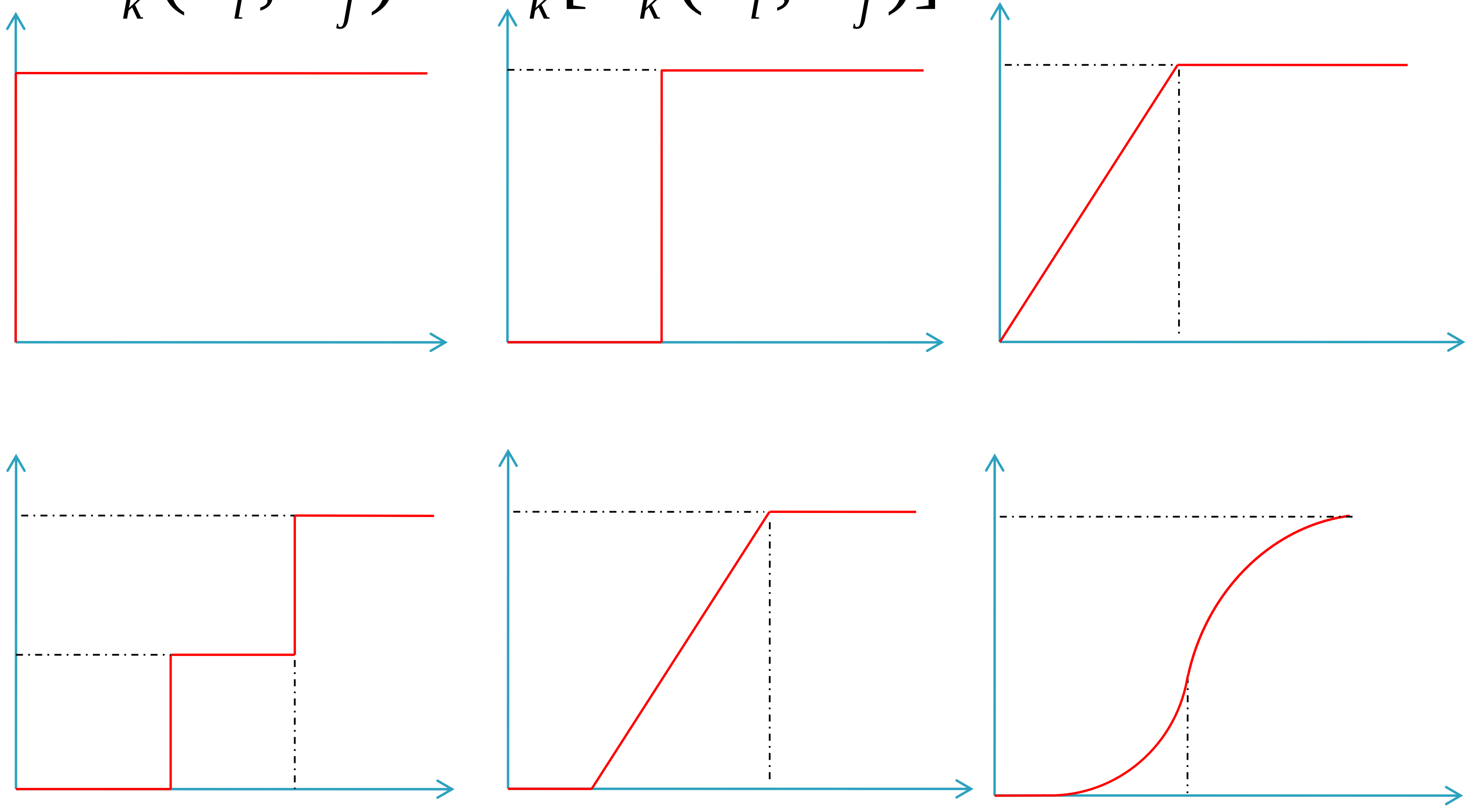
$$F=\{f_1,f_2,..,f_q\}$$

- W.l.g. these criteria have to be **maximized**

Step 1: uni-criterion preferences

$$\forall a_i, a_j \in A : d_k(a_i, a_j) = f_k(a_i) - f_k(a_j)$$

$$\pi_k(a_i, a_j) = P_k[d_k(a_i, a_j)]$$



Step 2: Compute preference matrix

$$\forall a_i, a_j \in A : \pi(a_i, a_j) = \sum_{k=1}^{q} w_k \pi_k(a_i, a_j)$$

As a consequence:

$$\pi(a_i, a_i) = 0$$

$$\pi(a_i, a_j) \geq 0$$

$$\pi(a_i, a_j) + \pi(a_j, a_i) \leq 1$$

Step 3: compute flow scores

$$\Phi^{+}(a_i) = \frac{1}{n-1} \sum_{a_j \in A} \pi(a_i, a_j)$$

$$\Phi^{-}(a_i) = \frac{1}{n-1} \sum_{a_j \in A} \pi(a_j, a_i)$$

$$\Phi(a_i) = \Phi^{+}(a_i) - \Phi^{-}(a_i)$$

Maximum number of parameters: **3.q-1**

PROMETHEE II

Complete ranking based on the net flow score.

$$a_i P a_j \Leftrightarrow \Phi(a_i) > \Phi(a_j)$$

$$a_i I a_j \Leftrightarrow \Phi(a_i) = \Phi(a_j)$$

PROMETHEE I

Partial ranking based on both the positive and negative flow scores.

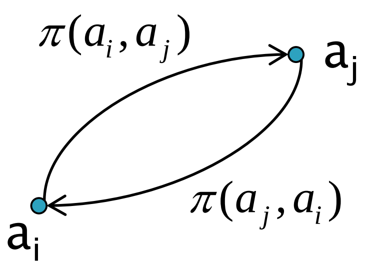
$$a_i P a_j \Leftrightarrow \Phi^+(a_i) > \Phi^+(a_j) \wedge \Phi^-(a_i) < \Phi^-(a_j)$$

$$a_i I a_j \Leftrightarrow \Phi^+(a_i) = \Phi^+(a_j) \wedge \Phi^-(a_i) = \Phi^-(a_j)$$

$$a_i J a_j \Leftrightarrow otherwise$$

The net flow score: a recipe ?

- From **local** to **global** information !



$S_i = S(a_i), S(a_j) = S_j$

ill-defined problem

- One could expect that:

$$\pi_{ij} - \pi_{ji} \approx S_i - S_j$$

Property

"The PROMETHEE multicriteria net flow $\phi(a_i)$ is the centred score s_i (i=1,…,n) that minimizes the sum of the squared deviations from the pair-wise comparisons of the actions"

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\left(s_i - s_j \right) - \left(\pi_{ij} - \pi_{ji} \right) \right]^2$$

Proof (1):

$$L\left(s_1,\ldots,s_n,\lambda\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(s_i-s_j\right)-\left(\pi_{ij}-\pi_{ji}\right)\right]^2-\lambda\sum_{i=1}^{n}s_i$$

$$\frac{\partial L\left(s_1,\ldots,s_n,\lambda\right)}{\partial s_i}=0$$

$$\frac{\partial L\left(s_1,\ldots,s_n,\lambda\right)}{\partial \lambda}=0$$

Proof (2):

$$L(s_1,\ldots,s_n,\lambda)=\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(s_i-s_j\right)-\left(\pi_{ij}-\pi_{ji}\right)\right]^2-\lambda\sum_{i=1}^{n}s_i$$

$$\frac{\partial L}{\partial s_i}=2\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\partial}{\partial s_i}\left\{\left[\left(s_i-s_j\right)-\left(\pi_{ij}-\pi_{ji}\right)\right]^2\right\}-\lambda$$

$$=4\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\left(s_i-s_j\right)-\left(\pi_{ij}-\pi_{ji}\right)\right]-\lambda$$

$$=4\left\{(n-1)s_i-\sum_{\substack{j=1\\ j\neq i}}^{n}s_j-\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\pi_{ij}-\pi_{ji}\right)\right\}-\lambda$$

$$=4\left\{ns_i-\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\pi_{ij}-\pi_{ji}\right)\right\}-\lambda=0$$

$$s_i=\frac{1}{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(\pi_{ij}-\pi_{ji})=\frac{n-1}{n}\phi(a_i)$$

A few words about rank reversal

Rank reversal

- We could have:

$$\pi_{ij} \geq \pi_{ji} \wedge \phi(a_i) \leq \phi(a_j)$$

- In other words: a pairwise rank reversal …
- This opens a discussion about rank reversal …
  - AHP: *Belton and Gear (1983), Saaty and Vargas (1984), Triantaphyllou (2001), Wang and Elhag (2006), Wijnmalen and Wedley (2009)*
  - ELECTRE: *Wang and Triantaphyllou (2005)*
  - PROMETHEE: *De Keyser and Peeters (1996)*
- The concept of rank reversal is not fully formalized (*add a copy of an alternative, deletion of a non discriminating criterion, deletion of an alternative, …)*
- *A direct consequence of Arrow's theorem*

Deletion of a non discriminating criterion

$$\phi_k(a_i) = 0 \forall a_i \in A$$

$$\phi(a_i) = \sum_{j=1}^{q} w_j \phi_j(a_i)$$

$$= \sum_{j=1, j \neq k}^{q} w_j \phi_j(a_i)$$

$$= W_k \sum_{j=1, j \neq k}^{q} \frac{w_j}{W_k} \phi_j(a_i)$$

$$= W_k \sum_{j=1, j \neq k}^{q} w'_j \phi_j(a_i)$$

$$= W_k \phi'(a_i)$$

	f_1	f_2	…	f_k	…	f_q
a_1	$f_1(a_1)$	$f_2(a_1)$	…	α	…	$f_q(a_1)$
a_2	$f_1(a_2)$	$f_2(a_2)$	…	α	…	$f_q(a_2)$
…	…	…	…	…	…	…
a_n	$f_1(a_n)$	$f_2(a_n)$	…	α	…	$f_q(a_n)$

where $W_k = \sum_{j=1, j \neq k}^{q} w_j$ and $w'_j = \frac{w_j}{W_k}$

$$\phi(a_i) > \phi(a_j) \Leftrightarrow \phi'(a_i) > \phi'(a_j)$$

Dominance

- Let us assume that:

$$f_k(a_i) \geq f_k(a_j), \forall k = 1,..q$$

- Then:

$$\phi(a_i) = \frac{1}{n-1} \sum_{k=1}^{q} w_k \sum_{b \in A} \pi_k(a_i, b) - \pi_k(b, a_i)$$

$$\geq \frac{1}{n-1} \sum_{k=1}^{q} w_k \sum_{b \in A} \pi_k(a_j, b) - \pi_k(b, a_j) = \phi(a_j)$$

This result holds for any set A such that a_i, a_j $\in$A

More general result (1)

Notations: $A_x = A \backslash \{x\} \quad \Phi_x(a)$

No RR $\Leftrightarrow \big(\Phi(a) - \Phi(b)\big)\big(\Phi_x(a) - \Phi_x(b)\big) > 0$

$$\text{if} \quad \Phi(a) - \Phi(b) > \frac{[(\pi_{ax} - \pi_{xa}) - (\pi_{bx} - \pi_{xb})]}{n-1}$$

No RR (for any action removed) if

$$\Phi(a) - \Phi(b) > \frac{\max_x[(\pi_{ax} - \pi_{xa}) - (\pi_{bx} - \pi_{xb})]}{n-1}$$

More general result (2)

➔ RR can only occur if

$$\Phi(a) - \Phi(b) < \underbrace{\frac{\max_x[(\pi_{ax} - \pi_{xa}) - (\pi_{bx} - \pi_{xb})]}{n-1}}_{\text{refined threshold (depends on the sample and (a,b))}} \leq \underbrace{\frac{2}{n-1}}_{\text{rough threshold (constant)}}$$

Generalization: when k actions are removed

No RR if $\Phi(a) - \Phi(b) > \dfrac{2k}{n-1}$

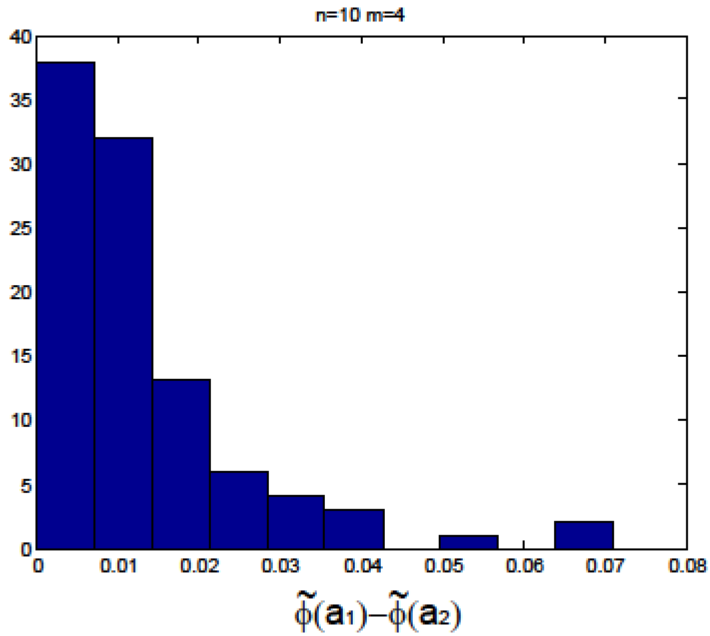
More general result (3)

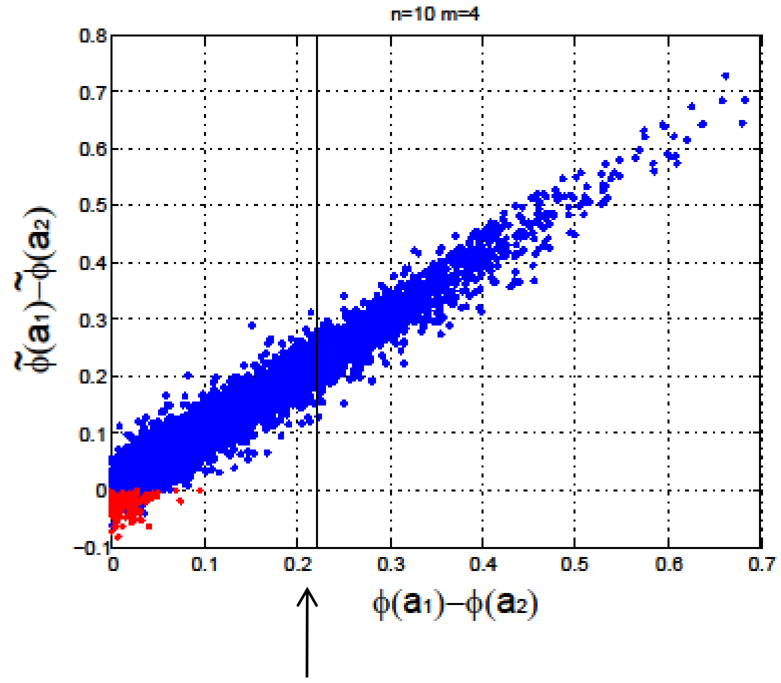
Statistical results relative to the «rough threshold» (for q = 2, DA=Unif)

n	nb RR	$b = \frac{2}{n-1}$	nb $\Delta\Phi \leq b$	nb RR $\mid \Delta\Phi \leq b$
5	2,20 %	0,50	47,4 %	4,6 %
10	0,98 %	0,22	33,5 %	2,9 %
15	0,66 %	0,14	24,7 %	2,6 %
20	0,45 %	0,10	19,9 %	2,2 %
50	0,18 %	0,04	9 %	1,9 %

Conclusion: The number of RR occurences is really small.

More general result (4)





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Verly, C. and De Smet, Y « Some considerations about rank reversal occurrences in the PROMETHEE methods »

Related works for PROMETHEE I

- No rank reversal will happen between a_i and a_j if

$$|\phi^+(a_i) - \phi^+(a_j)| \geq \frac{1}{n-1}$$

$$|\phi^-(a_i) - \phi^-(a_j)| \geq \frac{1}{n-1}$$

GAIA

<u>G</u>eometrical <u>A</u>nalysis for <u>I</u>nteractive <u>A</u>ssistance

GAIA (1)

- We have:

$$\Phi(a_i) = \frac{1}{n-1}\sum_{b\in A}\sum_{k=1}^{q} w_k.\pi_k(a_i,b) - \frac{1}{n-1}\sum_{b\in A}\sum_{k=1}^{q} w_k.\pi_k(b,a_i)$$

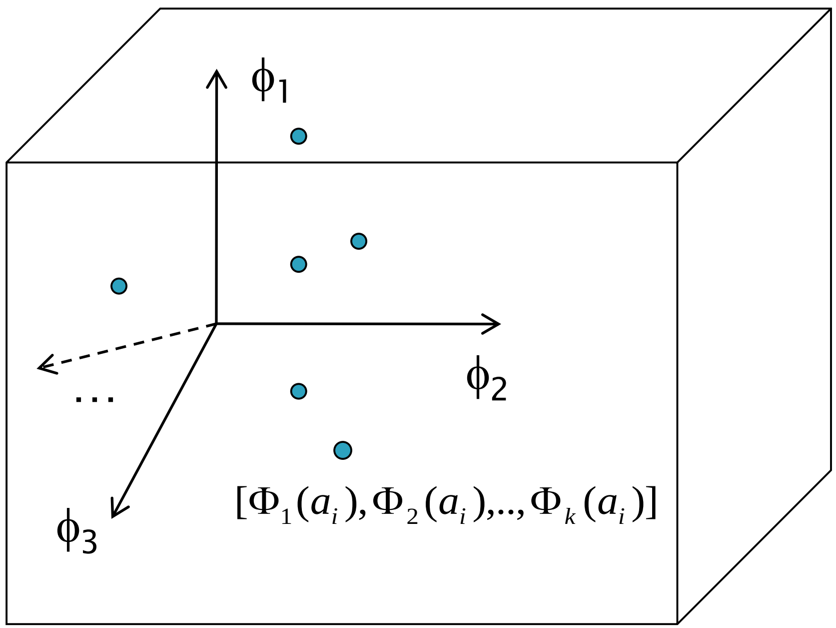
$$= \sum_{k=1}^{q} w_k.\frac{1}{n-1}\sum_{b\in A}\pi_k(a_i,b) - \pi_k(b,a_i) = \sum_{k=1}^{q} w_k.\phi_k(a_i)$$

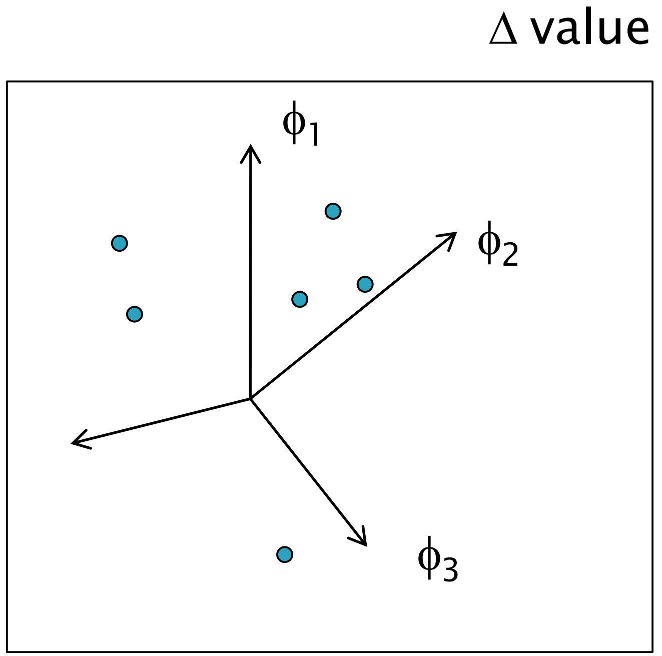
- Where $\Phi_k(a_i) = \frac{1}{n-1}\sum_{b\in A}\pi_k(a_i,b) - \pi_k(b,a_i)$

- In other words, every alternative can be represented by a vector:

$$[\Phi_1(a_i), \Phi_2(a_i), .., \Phi_k(a_i)]$$

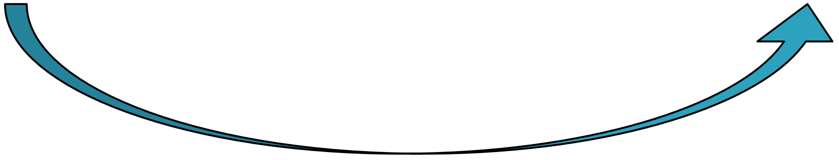
GAIA (2)





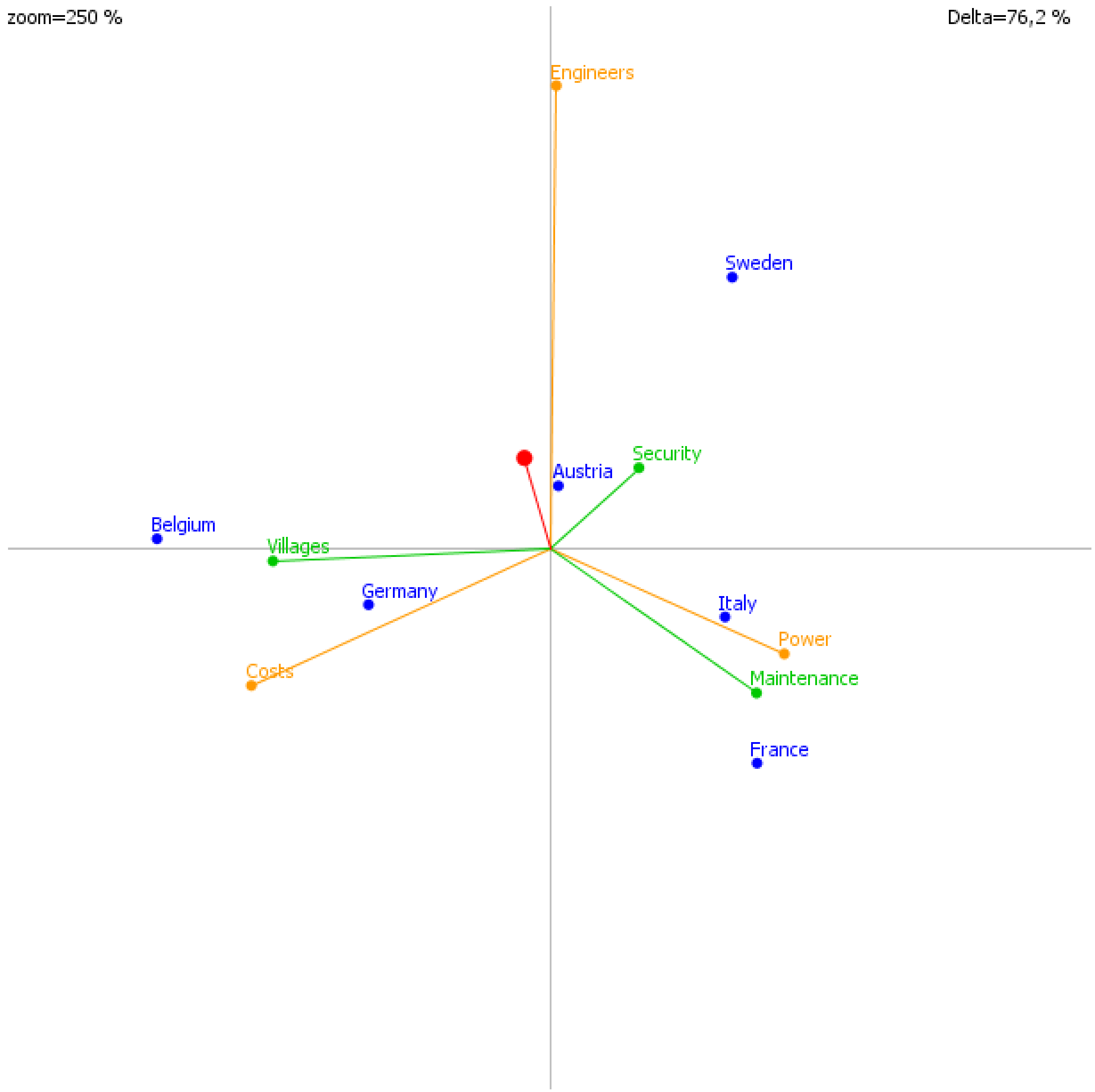
q dimensions

2 dimensions



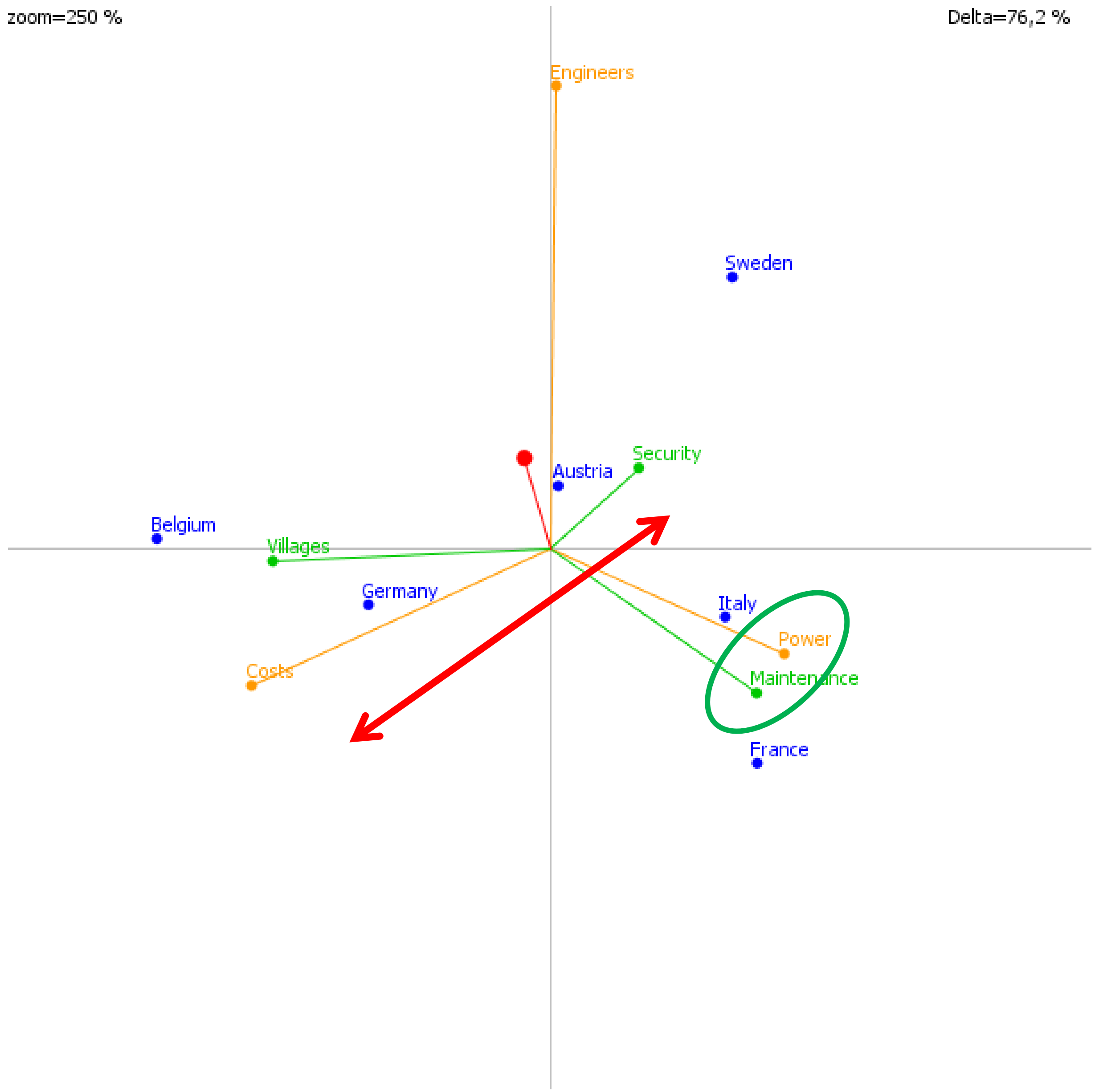
Principal component analysis

GAIA(3)

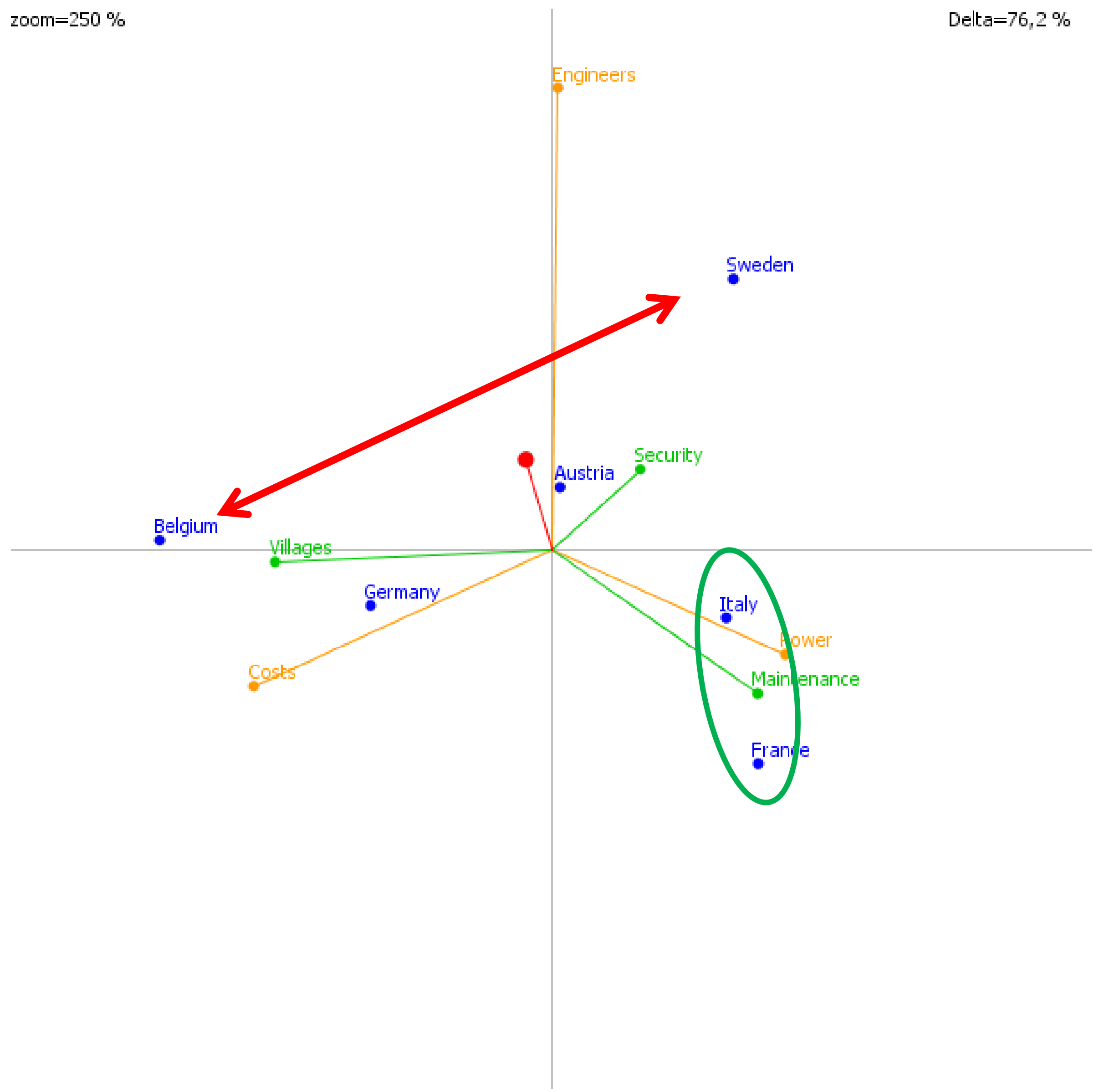


zoom=250 %
Delta=76,2 %
Engineers
Sweden
Security
Austria
Belgium
Villages
Germany
Italy
Power
Costs
Maintenance
France

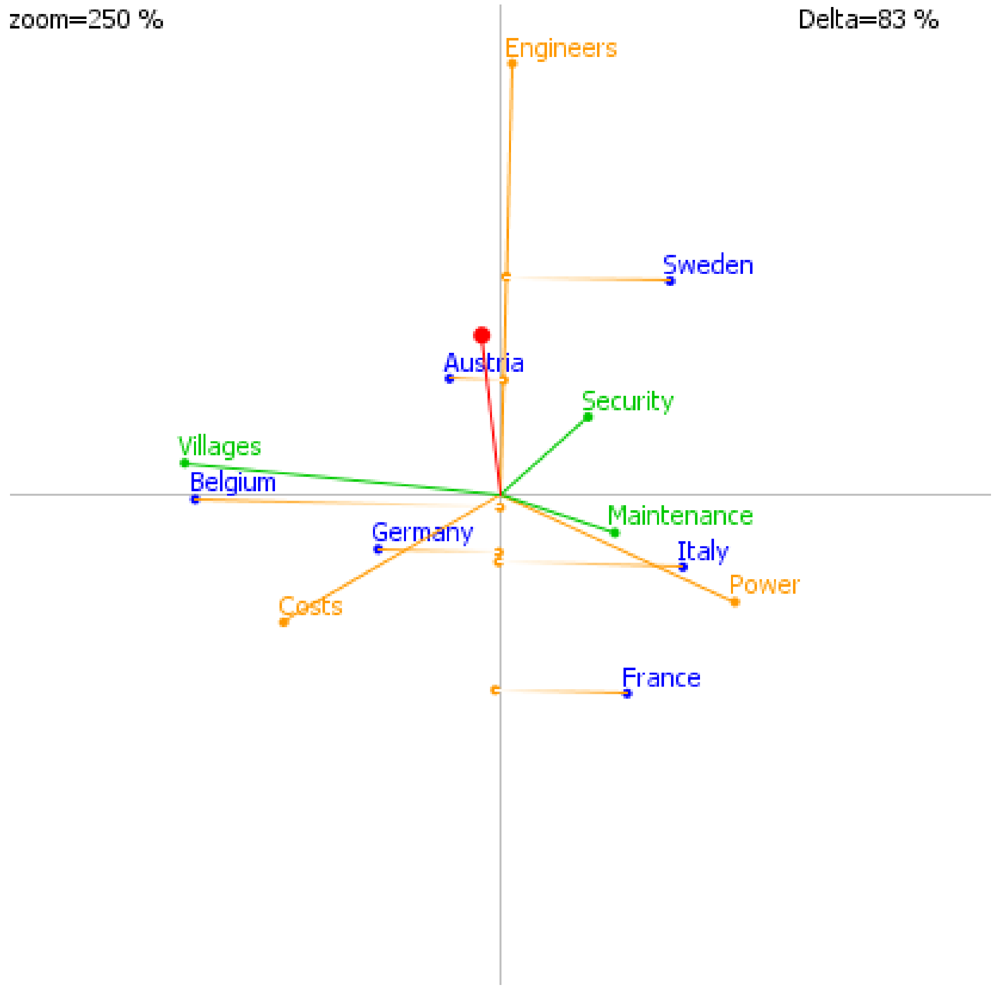
GAIA(4): criteria



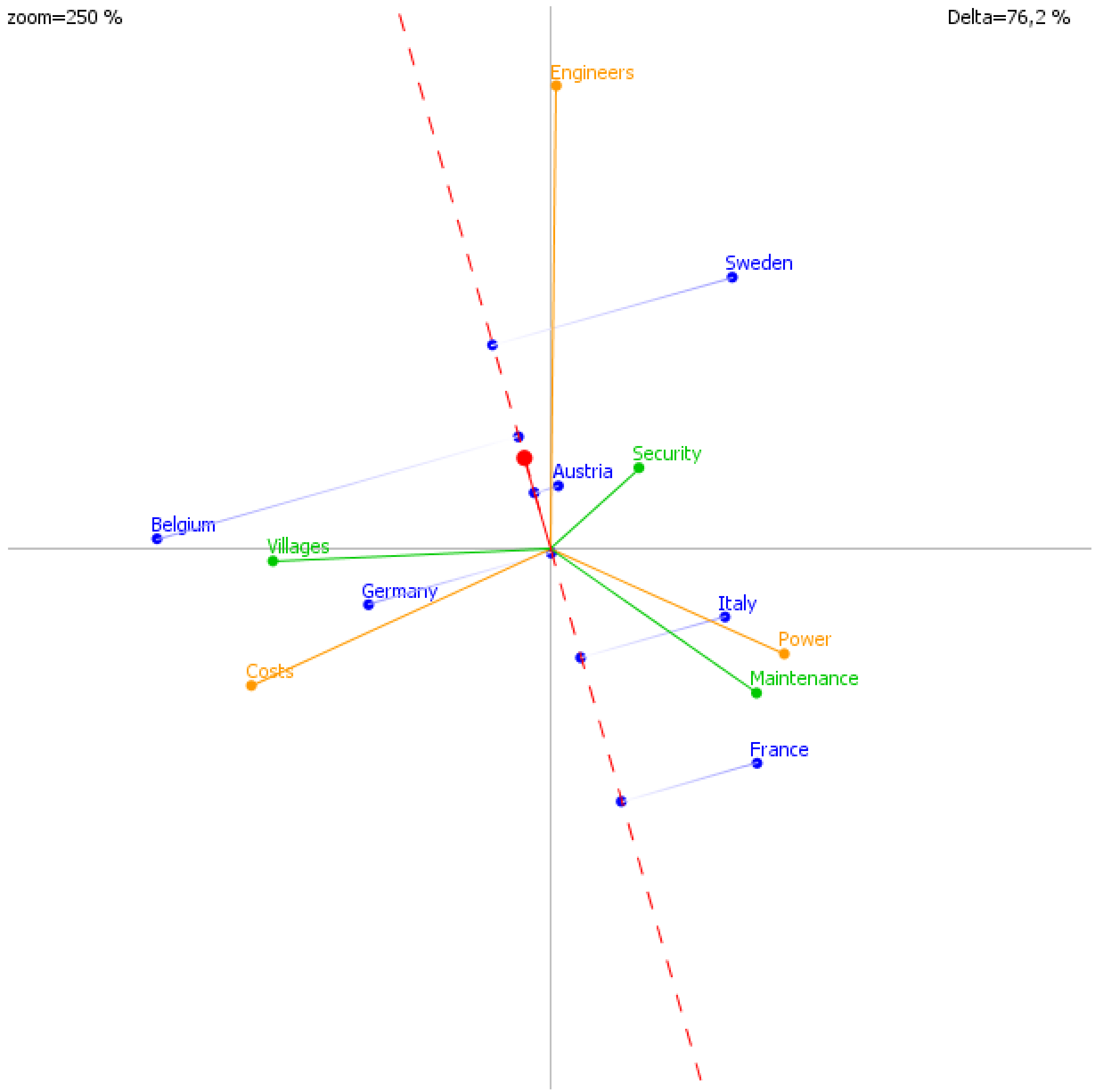
GAIA(5): alternatives



GAIA(6): alternatives / criteria



GAIA(7): Decision stick



Software demonstration