Introduction to Multi-Objective Optimization (MOO) and Multicriteria Decision Aid (MCDA)

Quantitative decision making ?

- Operational Research (OR)
- Few examples:
  - Knapsack;
  - Travelling salesman problem
  - Plant location problem
  - ...
- Main trend:

$$\text{opt } f(x)$$
$$x \in A$$

Garbage management

Everyday two cities, called X and Y, are producing respectively 500 tons and 400 tons of garbage. This garbage has to be incinerated in two incinerators denoted I1 and I2. The maximum daily capacity of I1 is 500 tons. For I2, it is 400 tons a day. The incineration cost per ton is 40 € for I1 and 30 € for I2. Burning ashes represents 20% of the total garbage and has to be placed in two garbage dumps denoted D1 and D2. Each garbage dump has a maximum capacity of 200 tons a day.

The transportation cost is 3 € per ton and kilometer. The distances between the cities, the incinerators and the garbage dumps are listed in the following table:

	I1	I2	D1	D2
X	30	5	5	8
Y	36	5	9	6

Knapsack problem

A climber can put up to 16 kg of supplies in his backpack. He can choose a number of units of three different products. The unit weight and the energy unit of these products are known. What does he need to take in order to maximize the total value in calories without exceeding 16 kg ?

	A	B	C
Weight (kg)	2	5	7
Value (calories)	4	10	15

Hitchcock Problem

Given 4 warehouses A, B, C, D containing quantities 40, 20, 30, and 55 of goods and 4 clients requesting quantities 12, 15, 30, 25 and knowing all the unit costs of transport from each warehouse to each customer, how do we serve all customers while minimizing the total cost ?

	1	2	3	4	Quantities
A	0.1	0.3	0.2	0.1	**40**
B	0.2	0.1	0.2	0.2	**20**
C	0.3	0.1	0.1	0.4	**30**
D	0.2	0.3	0.3	0.1	**55**
Demand	**12**	**15**	**30**	**25**	

Resources allocation

In an enterprise, four products share the same resources in equipments (machine hours), manual labor, and raw materials. The company has 240 hours of machine time available, 150 hours of manual labor, and 2 tons of raw materials. The profit per product is respectively of 5€, 6.5€, 8€, and 9€. The following table gives the requirements of all four products :

	A	B	C	D	Available
Machine hours	3	6	3	9	**240**
Manual labor	12	15	9	12	**150**
Raw materials	0.1	0.1	0.1	0.1	**2**
Profit (€)	**5**	**6.5**	**8**	**9**	

Crew resource management

- A new airline company requires different numbers of air-hostesses for each day. The number of cabin crew required for each day is given in the following table:

Days	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Min nb of cabin crew	27	23	25	29	24	26	21

- According to the relevant labor law, each cabin crew member must work five consecutive days and then receive two days off. For example, an cabin crew member who works from Monday to Friday must be off on Saturday and Sunday.
- The airline company wants to meet its daily requirements using only full-time cabin crew. Its objective is to minimize the number of full-time cabin crew that must be hired.

Maximum flow problem

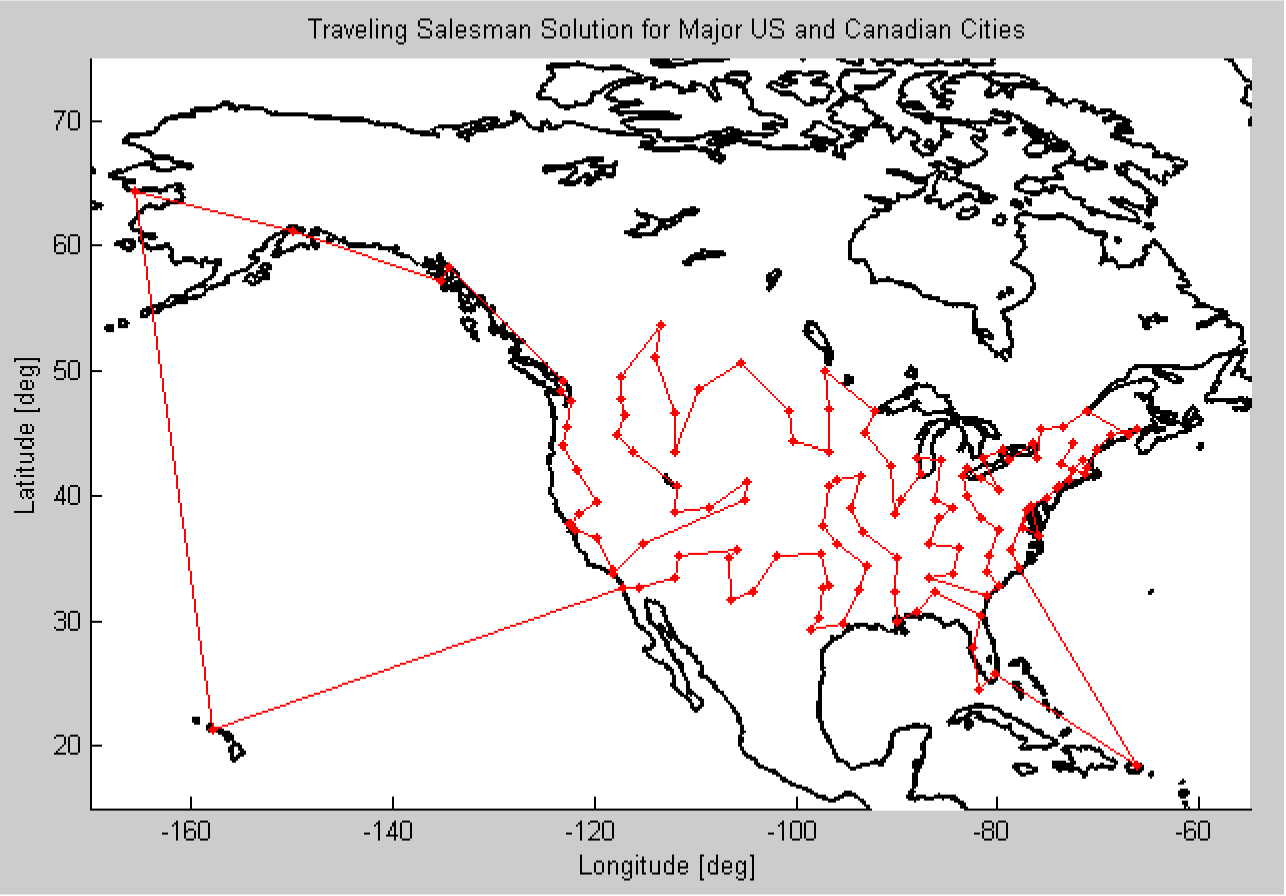
To find feasible flow through a network that is maximum (we assume a single-source and a single-sink).

Tasks allocation

M students have to be allocated to m individual projects. Each one has expressed his preferences on a 5-grades scale for each project (p_{ij} denotes the preference value of student I for project j).

a) How to find the allocation that maximizes the sum of preferences ?
b) How to find the allocation that maximizes the worst allocation ?

Travelling salesman problem (1)



Source: MATLAB

Travelling salesman problem (2)

- Combinatorial explosion ???
- 52 locations (cf U.S.A.), we have about 10^{69} solutions !
- Current computer: **3 Gigahertz** (= 3.000.000.000 operations per seconds)
- Reduced to $1 cm^3$ and 1.000.000 more powerfull
- Earth surface: 510.100.000 km^2
- Min distance moon-earth: 356.000 km
- 600 000 000 000 000 years

Cutting problem

A company have a stock of iron rails. The length of each rail is equal to 1m. One has to provide 36 rails of 28 cm and 24 rails of 45 cm. Three possible cuts are possible:

- 3 X 28 cm;
- 2 X 45 cm;
- 45 cm + 28 cm;

One looks to optimise the cuts in order to minimize the total lenght of scraps ?

Distribution centers

A company wants to build m production centers. There are n possible locations (n>m). The unit transport cost from location i to location j is denoted c_{ij}. The volume of goods to be shipped from production center k to production center l is denoted d_{kl} . Where should we build the production centers in order to minimize to total transport cost ?

The advantages of OR:

- The problem is well defined from a **mathematical point of view**
- **Total order** ⇔ alternatives can be ranked from the worst to the best ones
- Optimal solutions !
- **Huge developements**: mathematical programming, queueing theory, scheduling, discrete optimization, …

Some limits ?

- The set of alternatives **is known and does not evolve**
- No **uncertainty** about the evaluation
- **One criterion** (total order $\Rightarrow$ completeness, transitivity)
- Most strategic decision problems involve several criteria (remeber the introductory examples)
- All the criteria can not be monetarized (what is the price of a human life ?)

An alternative model

$$\text{opt } f_1(x), f_2(x), \ldots, f_q(x)$$
$$x \in A$$

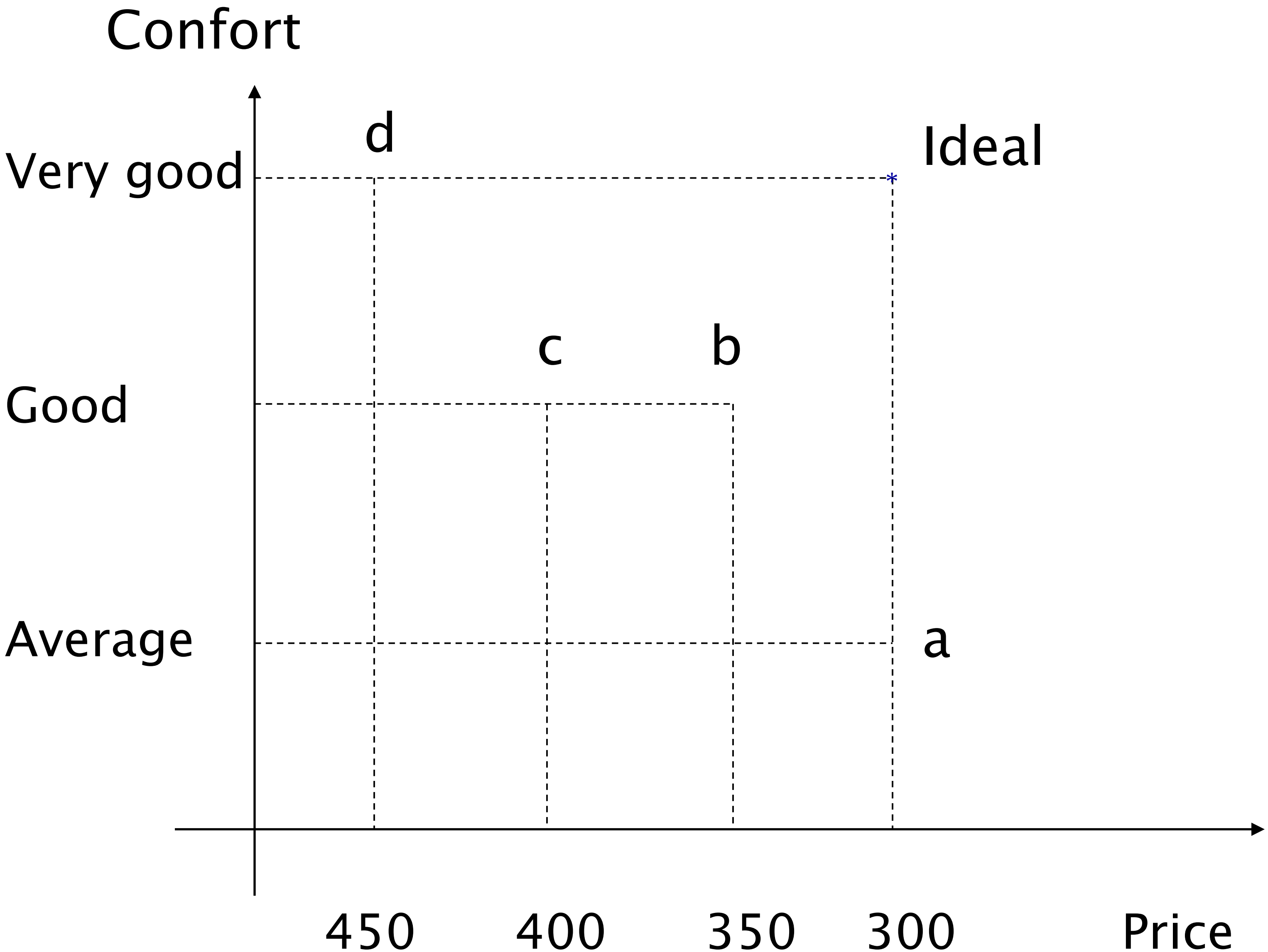
- Ill-defined problem – **<u>no optimal solution</u>**
- More realistic !

Multicriteria problems

	Criterion 1	Criterion 2	Criterion 3	…	Criterion q
Item 1	100	Medium	Yes	…	8
Item 2	87	Bad	No	…	9
Item 3	96	Good	Yes	…	10
Item 4	74	Very Good	No	…	7
…	…	…	…	…	…
Item n	55	Very bad	Yes	…	6

- **Main goal**: to rank the items
- Conflicting criteria
- Different units / scales
- **Ill-defined problem**: no optimal solution

Illustration




VOTING THEORY
MCDA / MOO

Basic multicriteria problem

	Crit. 1 (unit)	Crit. 2 (unit)	Crit. 3 (unit)	Crit. 4 (unit)	…
Action 1					
Action 2					
Action 3					
Action 4					
Action 5					
…					

Plant location

	Investment	Op. costs	Social acceptance	…
Site 1	18	135	G	…
Site 2	9	147	M	…
Site 3	15	129	VG	…
Site 4	12	146	VB	…
Site 5	7	121	G	…
…	…	…	…	…

Procurement

	Price	TBF	Maintenance	...
Product A	18	135	G	...
Product B	9	147	M	...
Product C	15	129	VG	...
Product D	12	146	VB	...
Product E	7	121	B	...
...	...	...	...	...

Link with voting theory

	Voter 1	Voter 2	Voter 3	…	Voter q
Cand.1	1	3	1	…	7
Cand. 2	4	2	4	…	2
Cand. 3	2	4	3	…	5
Cand. 4	7	7	5	…	1
…	…	…	…	…	…
Cand. n	3	1	8	…	2

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Common features ?

- Criterion = voter (total ranking)
- Alternatives = candidates
- Unanimity, Monotonicity, Independance to third alternatives, …

Main differences ?

- A given criterion **might be more important** than another (≠ anonimous voters)
- A criterion may give more information than a total ranking (different scales)
- The criteria are not imposed… what should be taken into account ?
- **<u>Conclusion</u>**: the voting problem was already complex (Arrow’s theorem), MCDA is even more …

MOO formulation

Objective**S**: Opt $z(x)$

$Z: \mathbb{R}^n \rightarrow \mathbb{R}^m$

Constraints: $g_j(x) \leq 0$

Variables: $x \in \mathbb{R}^n$

Example: bi-objective knapsack

- Project selection
- Budget constraint <= 900 k€
- Two objectives:
  - Maximize the expected profit
  - Maximize the number of employees
- Data:

	Project 1	Projet 2	Project 3	Project 4
Investment (k €)	200	300	400	500
Nb employees	2	3	5	6
Expected return (%)	2.5	4.5	4	2.5

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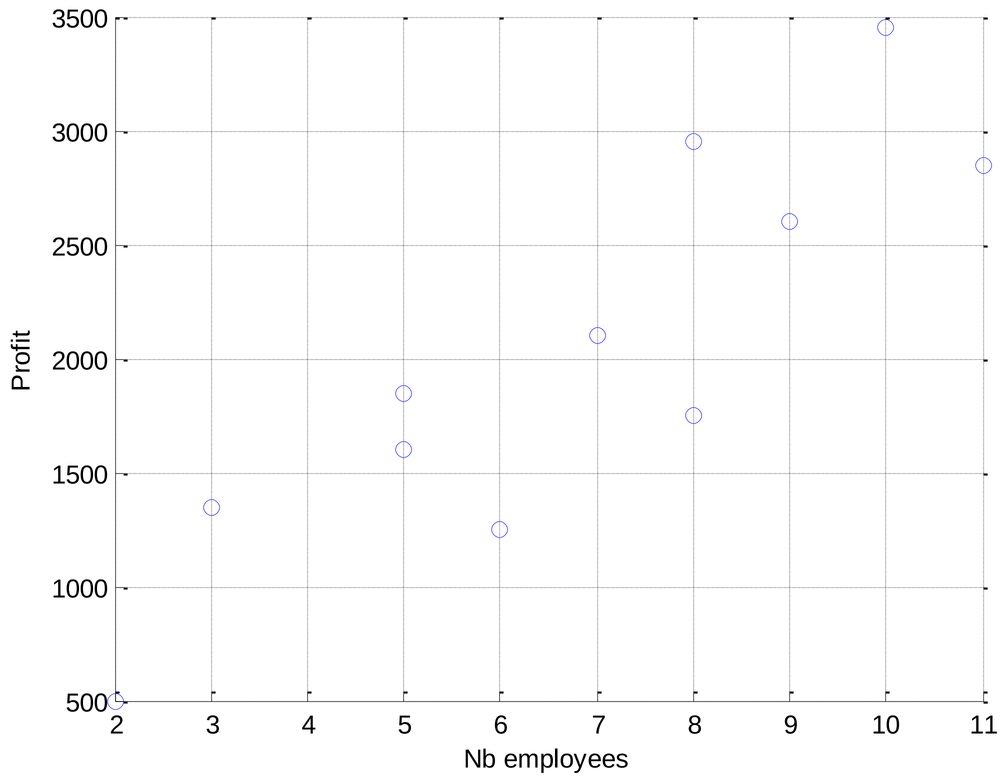


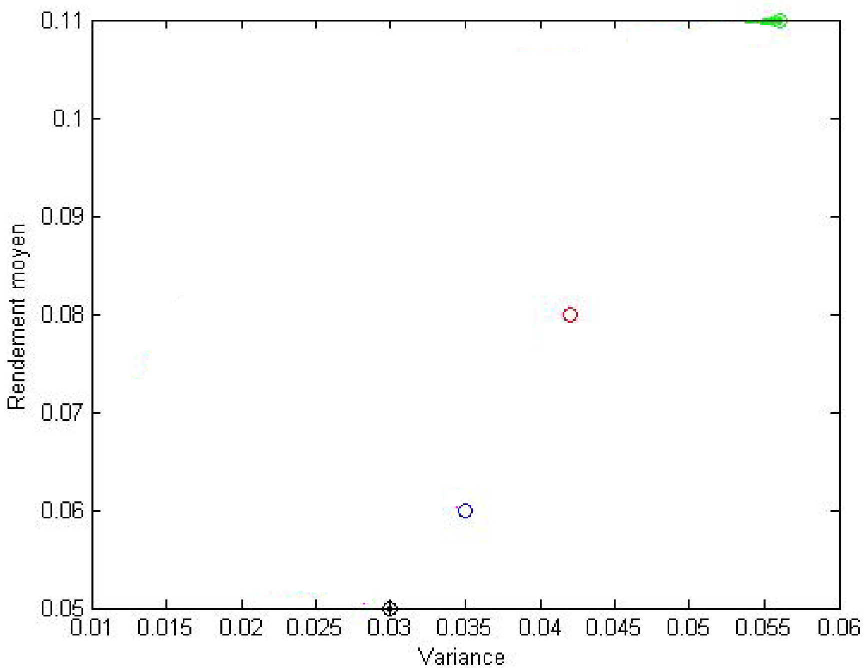
Example: bi-objective flowshop problem

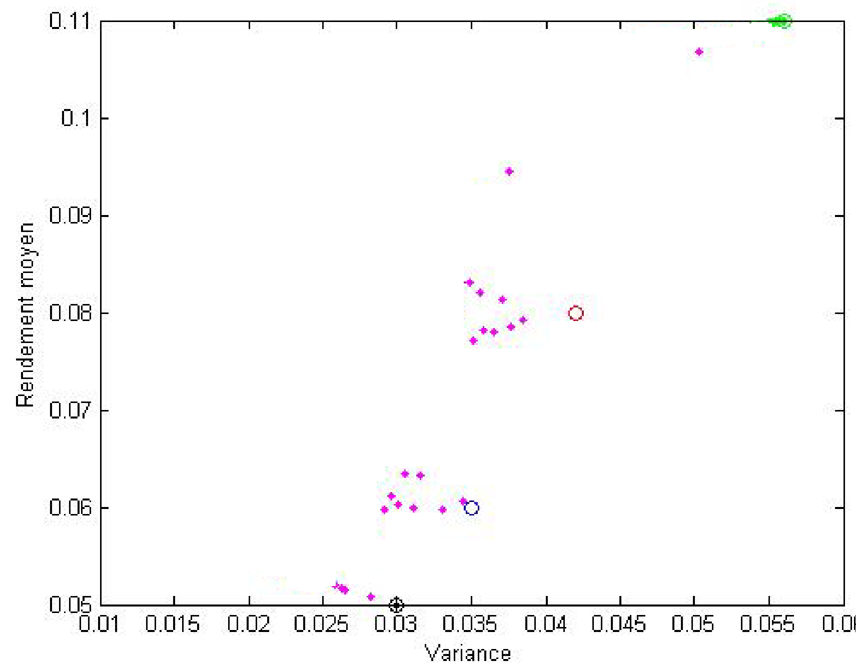
- M products
- N machines
- Every product has to pass on each machine (same sequence 1,2,3,..N)
- Different delays
- Different due dates
- Two objectives:
    - Minimize makespan
    - Minimize total tardiness

Portfolio management

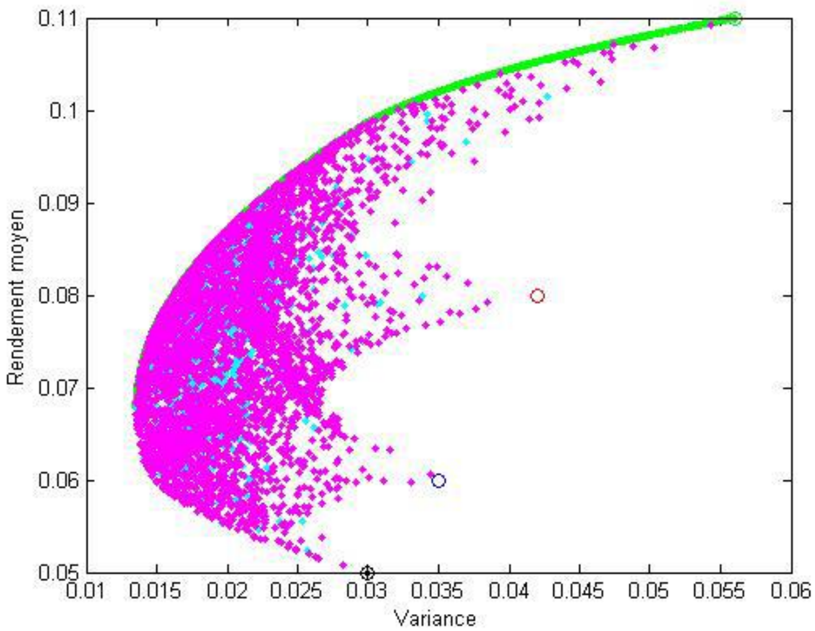
- A given capital K is invested into equities
- Bi-objective problem:
  - **Maximize** expected return !
  - **Minimize** risk !
- What are the “best” proportions ?

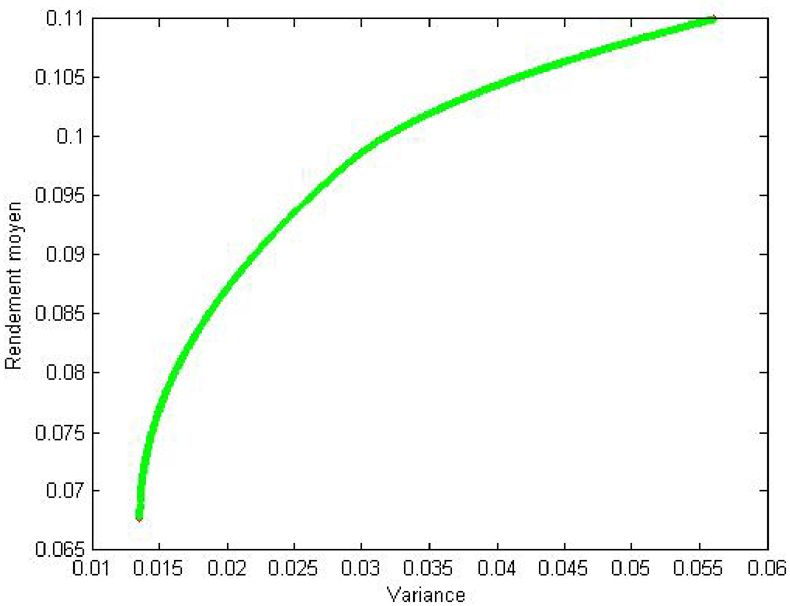
Examples with 4 investements (1)





Examples with 4 investements (2)





Back to the MOO formulation

ObjectiveS: Opt $z(x)$

$Z: \mathbb{R}^n \rightarrow \mathbb{R}^m$

Constraints: $g_j(x) \leq 0$

Variables: $x \in \mathbb{R}^n$

$x \in \mathbb{R}^n$ is called a solution

$X = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, \ \forall j\}$ is the feasible set

$x \in X$ is a feasible solution

$Y = \{z(x) | x \in X\}$ is the outcome set

$Y \in Y$ is a feasible point

Illustration on a bi-objective problem

- 2 cities, denoted X and Y
  - Tons of garbage: 100, 150
- 2 incinerators, denoted I_1 and I_2
  - Capacites: 150, 150
- Transportation costs per unit

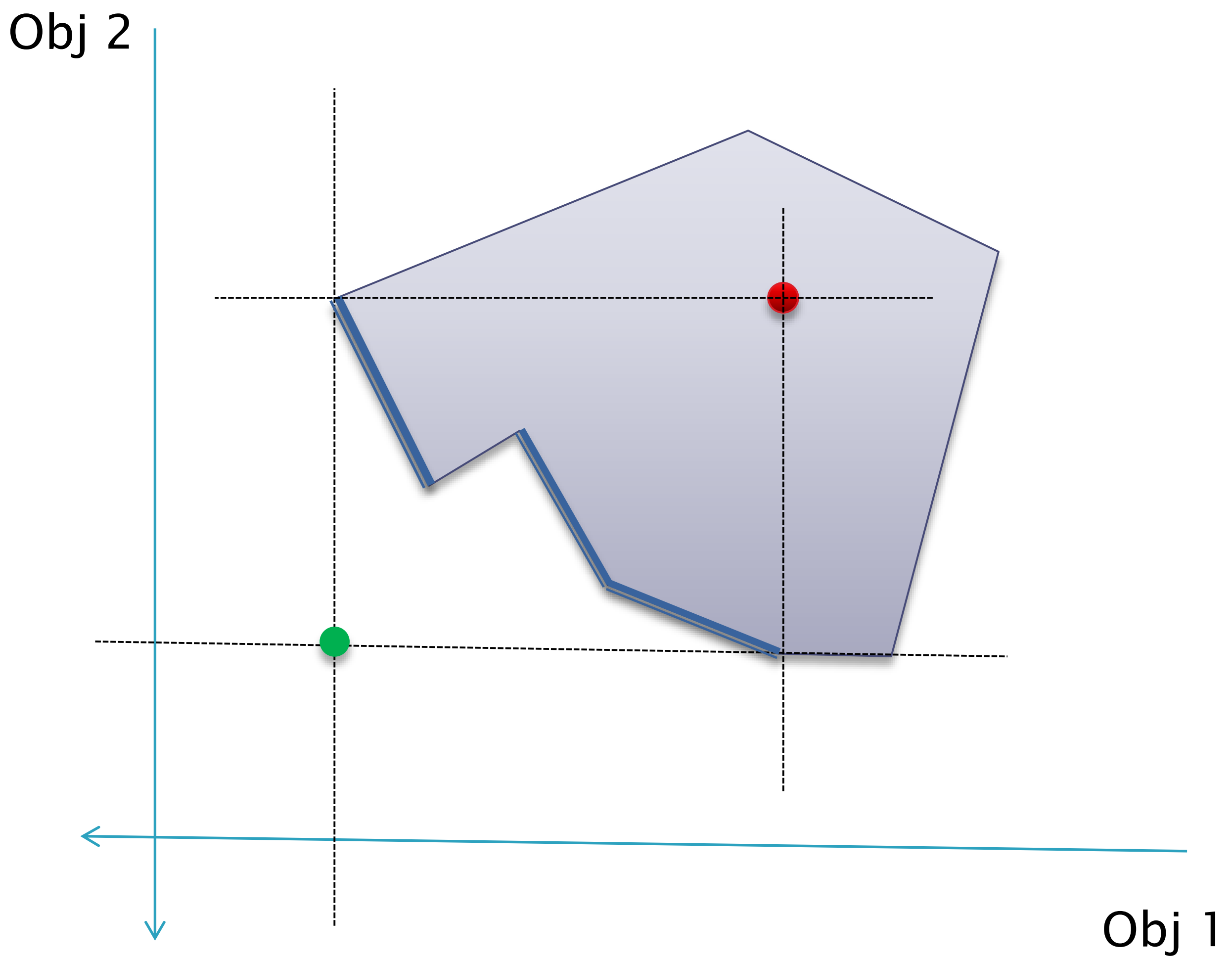
	I1	I2
X	2	3
Y	3	4

- Costs of incineration, 2 € (I_1), 1€(I_2)

Dominance principle

- Unanimity principle
- **<u>Definition</u>**: a is said to dominate b iff $f_i(a) \geq f_i(b)$ and $\exists j \mid f_j(a) > f_j(b)$
- A = {efficient solutions} ∪ {dominated solutions}
- PO(A) = Pareto optimal set = {efficient solutions}
- Main problem: #PO(A) ≈ #A
- The identification of PO(A) is often a problem itself …

Ideal and Nadir points

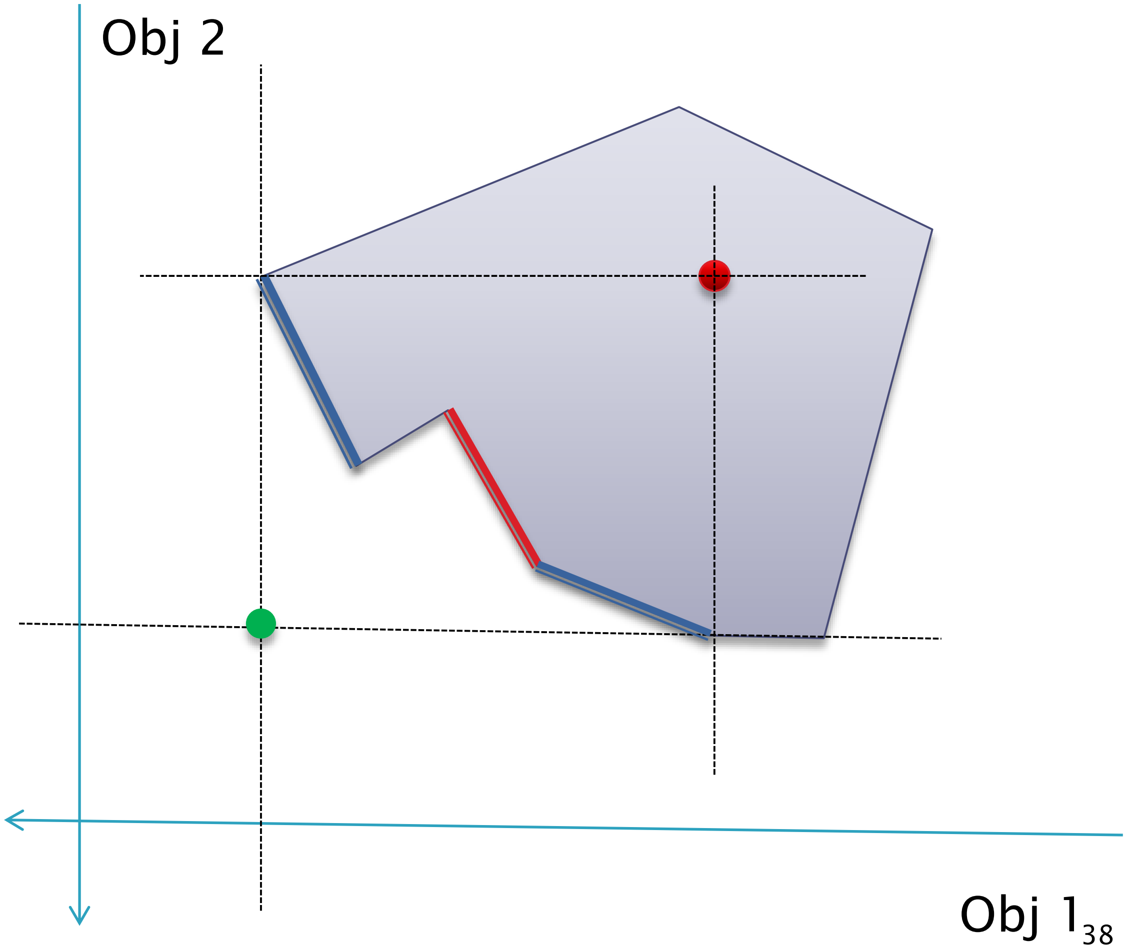


How to select the best solution ?

- Weighted sum?
- Weights ?

$$\min\{\sum_{k=1}^{m} w_k . z_k(x), x \in X\}$$

Unsupported solutions



A first approach:
the WEIGHTED SUM ?

A first approach: the weighted sum

	f_1	f_2		f_q
a_1				
a_2				
a_n				
	w_1	w_2		w_q

$$V(a) = \sum_{i=1}^{q} w_i . f_i(a)$$

a_i is better than a_j if $V(a_i) > V(a_j)$

The weighted sum is simple BUT it induces some effects on decision

The weighted sum: total compensation

- $V(a)=4,25$
- $V(b)=4$

	f_1	f_2	f_3	f_4
a	5	5	5	2
b	4	4	4	4
	1/4	1/4	1/4	1/4

The weighted sum: conflicts elimination

- $V(a)=V(b)=V(c)=V(d)=5$

	f_1	f_2
a	5	5
b	10	0
c	0	10
d	5	5
	1/2	1/2

The weighted sum: meaning of the weights

- « *Production is twice more important than quality* »

	Production (per month)	Quality	Score
a	100	100	100
b	120	80	106.66
Weights	2/3	1/3	

	Production (per week)	Quality	Score
a	25	100	50
b	30	80	46.66
Weights	2/3	1/3	

Solution ? Normalization ?

	f_1	f_2	f_3
a	2000	500	5
b	1360	440	10
c	1600	375	10
Weights	0.4	0.4	0.2

	f_1	f_2	f_3
a	1700	500	5
b	1360	440	10
c	1600	375	10
Weights	0.4	0.4	0.2

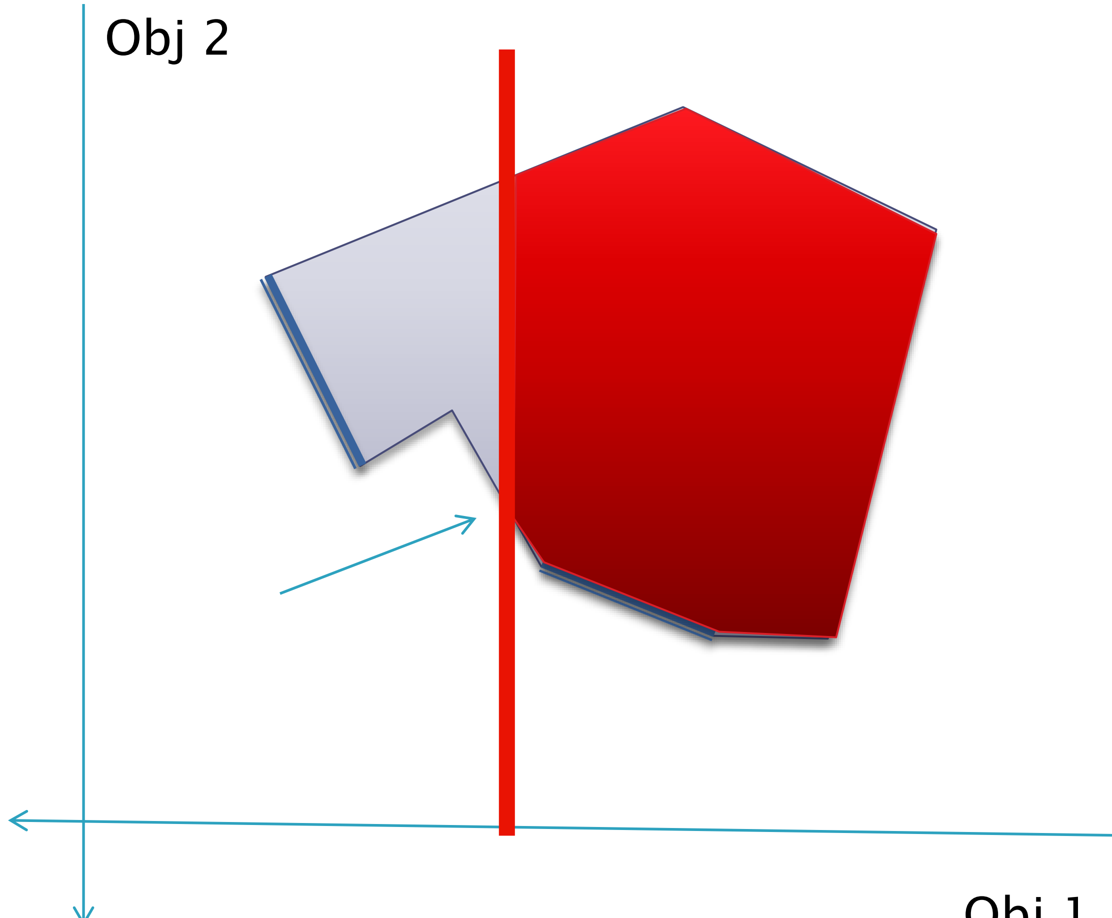
	f_1	f_2	f_3	Score
a	100	100	50	90
b	68	88	100	82.4
c	80	75	100	82

	f_1	f_2	f_3	Score
a	100	100	50	90
b	80	88	100	87.2
c	94	75	100	87.6

Epsilon constraint

- **Idea**: maximize one objective while considering others as constraints

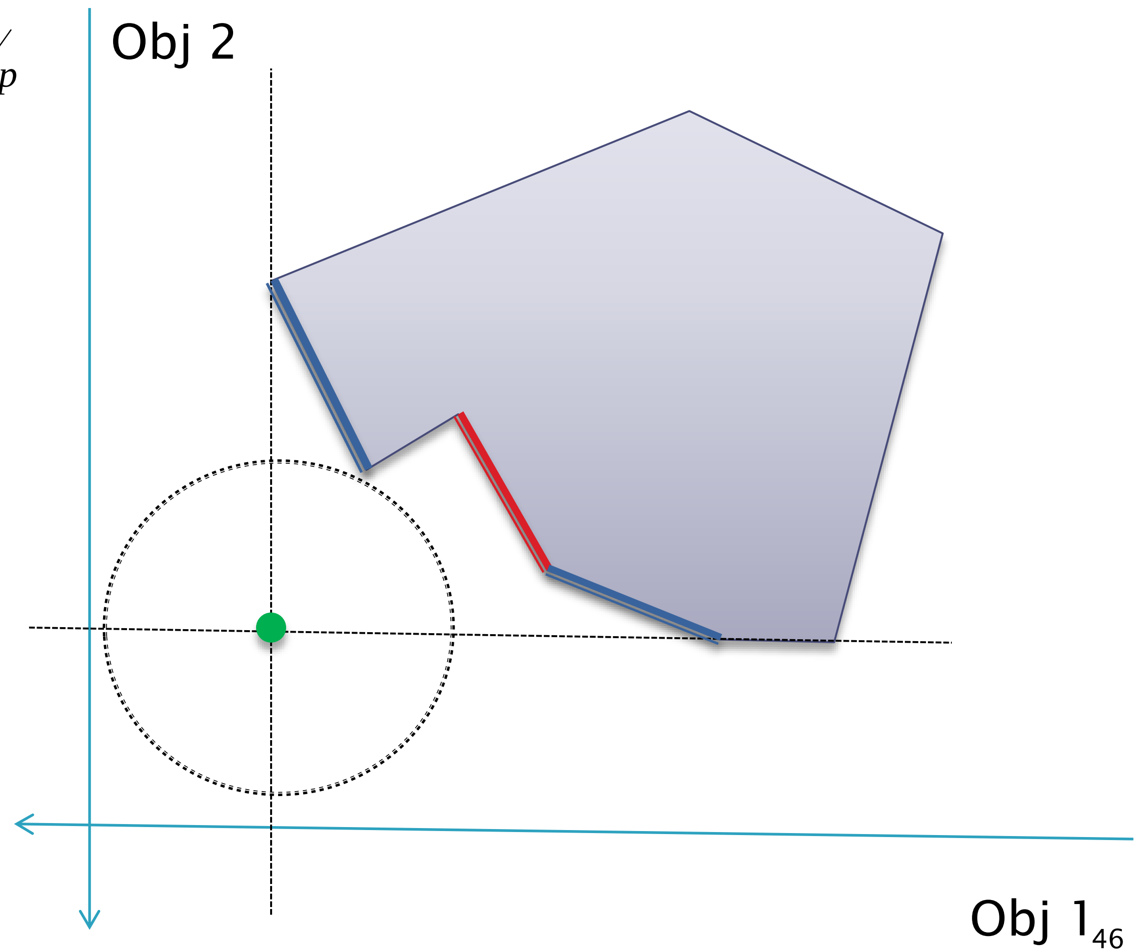
$$\min z_l(x)$$
$$z_k(x) \le b_k$$
$$x \in X$$



To be as close as possible to the ideal point?

- Weighted distance

$$\min\left(\sum_{k=1}^{m} w_k \left(z_k(x) - i_k\right)^p\right)^{1/p}$$



To conclude

- The main goal of Multi-objective optimization: **to identify the pareto optimal frontier**
- Hard problem
- Next step ? To select the final decision

Multicriteria Decision Aid