MATH-H-405 - Decision engineering

Solutions of Session 4: Game theory

Exercise 1

1. (R2, C2)
2. (R2, C3)
3. No
4. (R2, C3)

Exercise 2

1. (R1, C1) and (R2, C2)
2. (R2, C2)
3. (R1, C3) and (R2, C1)

Exercise 3
Also see Exercise 2.

	C_1	C_2
R_1	(1;1)	(0;0)
R_2	(0;0)	(1;1)

	C_1	C_2
R_1	(8;8)	(5;1)
R_2	(2;3)	(5;4)

Exercise 4
Let (s_1^*, s_2^*) be a solution obtained after successive removal of dominated strategies and let us suppose it is not a Nash equilibrium. Two cases are then possible:

1. $\exists s_1 | u_1(s_1^*, s_2^*) < u_1(s_1, s_2^*)$
2. $\exists s_2 | u_2(s_1^*, s_2^*) < u_2(s_1^*, s_2)$

Let us consider the first case (the second case can be demonstrated in the same way). The strategy (s_1, s_2^*) has been removed during the removal procedure, of line or a row. If we consider the removal of a line, we would have had $u_1(s_1^*, s_2^*) > u_1(s_1, s_2^*)$ which contradicts the hypothesis. If we consider the removal of a row, then the whole s_2^* strategy would have been removed and it would have not been possible to have (s_1^*, s_2^*) as a solution.
Let us now suppose that there exists another Nash equilibrium $(\tilde{s_1}, \tilde{s_2})$. This strategy has been removed during the removal procedure. If it has been removed during the removal

of a line, then there exists $s_1 \in S_1$ such that $u_1(s_1, \tilde{s_2}) > u_1(\tilde{s_1}, \tilde{s_2})$ which contradicts the definition of a Nash equilibrium. If it has been removed during the removal of a row, then there exists $s_2 \in S_2$ such that $u_2(\tilde{s_1}, s_2) > u_2(\tilde{s_1}, \tilde{s_2})$ which is also a contradiction.

Exercise 5

1. These are the condition of the "*equilibrium test*":
   - $\frac{\partial u_i(s_1^*,...,s_n^*)}{\partial s_i} = 0; \forall i = 1, 2, ..., n$ (condition for an optimum)
   - each s_i^* is the only stationary point of the function $u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n); s_i \in S_i$
   - $\frac{\partial^2 u_i(s_1^*,...,s_n^*)}{\partial s_i^2} < 0; \forall i = 1, 2, ..., n$ (condition for a maximum)
2. $$\begin{aligned} & \frac{\partial u_1(x,y)}{\partial x} = 0 \text{ and } \frac{\partial u_2(x,y)}{\partial y} = 0 \\ \Leftrightarrow \quad & y^2 - 2x = 0 \text{ and } 8 - 2xy = 0 \\ \Leftrightarrow \quad & (x, y) = (2, 2) \end{aligned}$$

Exercise 6

1. The payoff of a player k is the following expected utility function

$$\pi_k(b_1, \ldots, b_n) = \begin{cases} v_k - s & \text{if } b_k > s \\ 0 & \text{if } b_k < s \\ \frac{1}{r}(v_k - s) & \text{if } k \textit{ is among } r \textit{ buyers with highest bid} \end{cases}$$

   where s designates the second highest bid.

2. The strategy profile $(v_1, v_2, \ldots, v_n)$ is a Nash equilibrium for this game. We shall establish this in two steps.
   - *A player i never gains by bidding $b_i < v_i$*
     To see this, assume $b_i > v_i$ and let $b_{-i} = max_{j \neq i} b_j$. We distinguish five cases.
     CASE 1: $b_{-i} > b_i$
     In this case, some other bidder has the highest bid and so player i gets zero, which he could get by bidding v_i
     CASE 2: $v_i < b_{-i} < b_i$
     In this case, bidder i wins and gets $v_i - b_{-i} < 0$. However, if he would have bid v_i, then his payoff would have been zero (a higher payoff than that received by bidding b_i).

CASE 3: $b_{-i} = b_i$
Here bidder i is one among r buyers with the highest bid and he receives $\frac{v_i - b_{-i}}{r} < 0$. But, by bidding v_i, he can get 0, a higher payoff.
CASE 4: $b_{-i} < v_i$
In this case bidder i gets $v_i - b_{-i}$ which he could get by bidding v_i.
CASE 5: $b_{-i} = v_i$
Here again bidder i is one among r buyers with the highest bid and he receives $\frac{v_i - b_{-i}}{r} = 0$. But, by bidding v_i he can also get 0.

- *A player i never gains by bidding $b_i > v_i$*
  If $b_{-i} > v_i$ then bidder i would have a zero payoff which is the same as the payoff she would get if she bid v_i.

The strategy profile $(v_1, v_2, \ldots, v_n)$ is thus a Nash equilibrium. Therefore, it is reasonable to expect that every bidder will bid their true valuation of the painting and the bidder with the highest valuation wins. Note that this is true even if the bidder's do not know the valuation of the other bidders.

Exercise 7

1. From the definition of the Bertrand model, it is easy to see the quantity sold by each firm depending on the price they charge (their strategy). For the firm 1, we have (the same development applies for firm 2):

$$q_1 = \begin{cases} 0 & \text{if } p_1 > p_2 \\ A - p_1 & \text{if } p_1 < p_2 \\ \frac{A-p_1}{2} & \text{if } p_1 = p_2 \end{cases} \tag{1}$$

2. For the following, we will take the point of view of the firm 1 (the same development applies for firm 2).
   First, let us compute the utility for each firm which is just the difference between how much it will get from selling the product and how much it will cost to produce it:

$$\begin{aligned} U_1(p_1) &= (p_1 - c_1)q_1 \\ U_2(p_2) &= (p_2 - c_2)q_2 \end{aligned}$$

   To ensure that the quantity sold $q_1 \geq 0$, we have that $p_1 \leq A$.
   Also, to ensure that the utility $U_1 \geq 0$, we have that $p_1 \geq c_1$.
   So, the condition is to have $c_1 \leq p_1 \leq A$, and for the firm 2 we have $c_2 \leq p_2 \leq A$.

   Second, from point 1., we see that it is only interesting for the firm 1 to have its price $p_1 \leq p_2$ (otherwise $q_1 = 0$).

So, the condition becomes $c_1 \leq p_1 \leq p_2$, and for the firm 2 we have $c_2 \leq p_2 \leq p_1$.
Now, consider that $c_1 < c_2$.
In the worst case for firm 1, $p_1 = c_1$ so we can write that $p_1 = c_2 - \epsilon$. Since the condition for firm 2 says that $p_2 \geq c_2$, we have that $p_1 < p_2$ so firm 1 wins the market and sells $q_1 = A - c_2 + \epsilon$ and firm 2 sell nothing ($q_2 = 0$).
The case $c_2 < c_1$ will give similar conclusion but with firm 2 winning the market.
We thus see that there is an equilibrium only if $c_1 = c_2$ but since the hypothesis says that $c_1 \neq c_2$, there is no equilibrium for this game.