Introduction to game theory

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« Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact »

M.J. Osborne & A. Rubinstein

A first example

Suppose USAir and American Airlines (AA) are thinking about pricing a round-trip airfare from Chicago to New York. If both airlines charge a price of $500, the profit of USAir would be $50 million, and the profit of AA would be $100 million. If USAir charges $500 and AA charges $200, than the profit of AA is $200 million and USAir makes a loss of $100 million. If, however, USAir sets a price of $200 and AA charges $500, then USAir makes a profit of $150 million, while AA loses $200 million. If both charge a price of $200 than both airlines end up with losses of $10 million each. This information can be depicted in the following table:

	American Airlines		
USAir	Fare	$500	$200
	$500	(50,100)	(-100,200)
	$200	(150,-200)	(-10,-10)

Rational behavior

- A set **A** of actions from which the decision maker makes a choice
- A set **C** of possible consequences of these actions
- A consequence function $g: A \rightarrow C$ that associates a consequence with each action
- **A preference relation** (complete transitive reflexive binary relation) $\succsim$ on the set C
- The decision maker chooses the action a^* such that $g(a^*) \succsim g(a)\ \forall\ a \in A$

Often …

- The decision maker's preferences are specified by giving a utility function
- $U: C \rightarrow \Re$
- $g(a) \gtrsim g(b) \Leftrightarrow U(a) \geq U(b)$

Definition strategic game

A strategic game consists of:

- A finite set N (the set of players)
- For each player $i \in N$ a non empty set A_i (the set f actions available to player i)
- For each player $i \in N$ a preference relation $\succsim_i$ on $A = X_{j \in N} A_j$

An illustrative example:

	Player 2			
Player 1	Strategy	L	C	R
	T	(1,0)	(1,3)	(3,0)
	M	(0,2)	(0,1)	(3,0)
	B	(0,2)	(2,4)	(5,3)

Definition: A strategy s_i of player 1 is said to:

A. *dominate* another strategy s_j of player1 if

$$U_1(s_i, s) \geq U_1(s_j, s)$$

for each strategy s of player 2, and

B. *strictly dominate* another strategy s_j of player 1 if

$$U_1(s_i, s) > U_1(s_j, s)$$

for each strategy s of player 2.

Idea: Iterated elimination of strictly dominated strategies

	Player 2			
Player 1	Strategy	L	C	R
	T	(1,0)	(1,3)	(3,0)
	M	(0,2)	(0,1)	(3,0)
	B	(0,2)	(2,4)	(5,3)

R has been eliminated …

	Player 2		
Player 1	Strategy	L	C
	T	(1,0)	(1,3)
	M	(0,2)	(0,1)
	B	(0,2)	(2,4)

M has been eleminated…

		Player 2	
Player 1	Strategy	L	C
	T	(1,0)	(1,3)
	B	(0,2)	(2,4)

→ **(B,C)**

The prisoner's dilemma

Two individuals who have committed a crime have a choice of either confessing the crime or keeping silent. In case one of them confesses and the other keeps silent, then the one who has confessed does not go to jail, whereas the one who has not confessed gets a sentence of ten years. In case both confess, then each gets a sntence of five years. If both do not confess, then both get off fairly lightly with sentences of one year each.

The matrix game shows clearly that there are two players and the strategy set *of* each player is (Mum, Fink). The payoffs are given by tha pairs (a,b) for each outcome, with a being player 1's payoff and b player 2's payoff; here, of course, $-a$ and $-b$ represent years in jail. The matrix completely describes a game in strategic form.

	Player 2		
Player 1	Strategy	Mum	Fink
	Mum	(-1,-1)	(-10,0)
	Fink	(0,-10)	(-5,-5)

Nash equilibrium

- A Nash equilibrium of a strategic game $\langle N,(A_i),(\gtrsim_i)\rangle$ is a profile $a^* \in A$ such that:

$$(a^*_{-i}, a^*_i) \gtrsim_i (a^*_{-i}, a_i) \ \forall\, a_i \in A_i$$

Example: Battle of sex
A coordination game…

	Wife		
	Strategy	Bullfight	Opera
Husband	Bullfight	(2,1)	(0,0)
	Opera	(0,0)	(1,2)

Example: Battle of sex 2

	Wife		
	Strategy	Golf	Tennis
Husband	Golf	(2,2)	(0,0)
	Tennis	(0,0)	(1,1)

Example: Head or tail

	Player 2		
Player 1	Strategy	Head	Tail
	Head	(1,-1)	(-1,1)
	Tail	(-1,1)	(1,-1)

Example: Hack or dove

	Player 2		
Player 1	Strategy	Dove	Hawk
	Dove	(3,3)	(1,4)
	Hawk	(4,1)	(0,0)

The Cournot Duopoly Model

This is a strategic form game played between two firms; we will call them firm 1 and firm 2. The two firms produce identical products, with firm 1 producing an amount of q_1 units and firm 2 producing an amount of q_2 units. The total production by both firms will be denoted by q; that is, $q = q_1 + q_2$.

Let $p(q) = A - q$ be the price par unit of the product in the market, where A is a fixed number. Assume that the total cost to firm i of producing the output q_i is $c_i q_i$, where the c_i are positive constants.

This economic model may be written as a strategic form game in which:

- There are two players: the two firms.
- The strategy set of each player is the set of positive quantities that a firm can choose. That is, the strategy set of each player is $(0, \infty)$.
- The payoff function of firm i is simply its profit function

$$\Pi_i (q_1, q_2) = (A - q_1 - q_2)\, q_i - c_i\, q_i.$$

The Cournot Duopoly Model

The problem faced by the firms is how to determine how much each one of them could produce in order to maximize profit–notice that the profit of each firm depends on the output of the other firm. Since we will assume that the firms choose their production quantities independently and simultaneously, it is reasonable to think of the Nash equilibrium as the solution.

Two other applications

- The median voter theorem
- Second Price Sealed Bid auctions (Vickrey auctions)

References

- « A course in game theory », M.J. Osborne & A. Rubinstein
- « Games and Decision making », C. D. Aliprantis & S. K. Chakrabarti