Decision under risk and uncertainty

(A ridiculously sketchy introduction)

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Decision: typology

Decision "under certainty"

- A: set of alternatives (possible decisions)
- X: set of consequences
- $c(a) \in X$: consequence of implementing $a \in A$

Problem

- help someone compare alternatives in A on the basis of their consequences

Classic problems

- $|A|$ "large": combinatorial optimization, mathematical programming
- $x \in X$ such that $x = (x_1, x_2, \ldots, x_m)$: multicriteria problems

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Decision under risk and uncertainty

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- *la décision ne dispose que pour l'avenir* (cf. art. 2 of the French Civil Code:)
  - $c(a)$ is not known with certainty

Decision under risk

- $c(a)$ is a probability distribution on X

Decision under uncertainty

- $c(a)$ is known conditionally upon the occurrence of a number of "scenarios"

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Plan

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Decision under uncertainty

Context

- impossibility to determine with certainty the consequences of implementing an alternative
- no probability
- Nature decide of everything that is not under my control
- the consequences of my decisions depend upon my decisions and Nature's decisions ("states of Nature" or "scenarios")
- Nature does not care: dropping a slice of bread on the floor (the "tartine beurrée" exepriment)

Problem

- you must choose an alternative before knowing Nature's decision

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Model

Model

- A: set of alternatives. An element $a \in A$ is an alternative that can be implemented
- E: set of states of Nature. An element $e \in E$ is a decision that Nature can take and that can influence the consequences of at least one alternative in A
- X: set of consequences
- c: mapping from $A \times E$ to X

Decision table (finite case: m alternatives, n states)

Decision table

c	e_1	e_2	$\cdots$	e_i	$\cdots$	e_n
a_1	$c(a_1, e_1)$	$c(a_1, e_2)$	$\cdots$	$c(a_1, e_i)$	$\cdots$	$c(a_1, e_n)$
a_2	$c(a_2, e_1)$	$c(a_2, e_2)$	$\cdots$	$c(a_2, e_i)$	$\cdots$	$c(a_2, e_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_j	$c(a_j, e_1)$	$c(a_j, e_2)$	$\cdots$	$c(a_j, e_i)$	$\cdots$	$c(a_j, e_n)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_m	$c(a_m, e_1)$	$c(a_m, e_2)$	$\cdots$	$c(a_m, e_i)$	$\cdots$	$c(a_m, e_n)$

Remark

- obtaining such a “decision table” is a huge work in practice

Exemple: the omelette

The omelette

$A = \{\text{Bowl}, \text{Thrash}, \text{Aux. Bowl}\}$

$E = \{\text{Good}, \text{Bad}\}$

c	Good	Bad
Bowl	O. of 6	No O.
Thrash	O. of 5	O. of 5
Aux. Bowl	O. of 6 + Bowl to wash	O. of 5 + Bowl to wash

Remarks

- no probabilities
- tastes & beliefs
- possibility to acquire additional information (experimentation)

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Examples

Bank

	Default	$\overline{\text{Default}}$
Accept	...	...
Refuse	...	...
Accept with guarantees	...	...

New product

	Success	$\overline{\text{Success}}$
Launch	...	...
$\overline{\text{Launch}}$	...	...

Example

Example

$X = \mathbb{R}$, preference increases with the numbers (€)

c	e_1	e_2	e_3
a_1	40	70	−20
a_2	−10	40	100
a_3	20	40	−5

Classic criteria

- no information about the likelihood of the states of Nature
- no particular model for tastes

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Dominance

Definition

$a \in A$ (strictly) dominates $b \in A$ ($a \; D \; b$) if:

- $c(a, e) \geq c(b, e), \forall e \in E$,
- $\exists e \in E$ such that $c(a, e) > c(b, e)$

Remark

D is a transitive and asymmetric binary relation

Definition

$a \in A$ is efficient if it is not dominated by another alternative in A. When A and E are finite, the set of efficient alternatives $A^* \subseteq A$ defined by:

$$A^* = \{a \in A : \text{Not}[b \; D \; a], \forall b \in A\}$$

is always nonempty

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Dominance

Remarks

- $a \; D \; b \Rightarrow a \succ b$, whatever the likelihood of the sates of Nature
- in real-world problems: $A^* = A$
- limiting attention to A^* might not be adequate, e.g., if there are doubts on the feasibility of some alternatives in A. The set A^* might not contain "close contenders"
- same problems as in MCDA/MCDM

Example

c	e_1	e_2	e_3	e_4	e_5	e_6	$\ldots$ e_{100}
a	100	100	100	100	100	100	$\ldots$ 100
b	99	99	99	99	99	99	$\ldots$ 99
c	100.5	0	0	0	0	0	$\ldots$ 0
d	0	100.5	0	0	0	0	$\ldots$ 0

- $A = \{a, b, c, d\}$
- $A^* = \{a, c, d\}$ because $a\ D\ b$
- b is a "close contender"

Example

c	e_1	e_2	e_3	e_4	e_5	e_6	$\ldots$ e_{100}
a	100	100	100	100	100	100	$\ldots$ 100
b	99	99	99	99	99	99	$\ldots$ 99
c	100.5	0	0	0	0	0	$\ldots$ 0
d	0	100.5	0	0	0	0	$\ldots$ 0

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go faster

Remark

Every alternative that is solution of problem (P)

$$\max_{a \in A} \sum_{e \in E} p(e)c(a,e)$$

$$\text{s.t.}$$

$$\sum_{e \in E} p(e) = 1 \qquad (P)$$

$$p(e) > 0, e \in E$$

is efficient

Suppose that a is solution of (P) and that a is not efficient. Since $c(b,e) \geq c(a,e), \forall e \in E$ and $c(b,e') > c(a,e')$ we have

$$\sum_{e \in E} p(e)c(b,e) > \sum_{e \in E} p(e)c(a,e)$$

Remark

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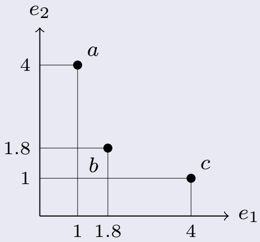
Converse



- $A = A^* = \{a, b, c\}$
- b cannot be solution of (P)

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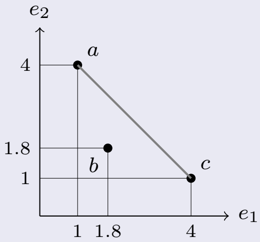

- $A = A^* = \{a, b, c\}$
- b cannot be solution of (P)

Wald's criterion (Max Min)

Idea

- extreme pessimism: base your choice on the worst situation (Max Min)
- choose any alternative $a \in A$ solution of:

$$\max_{a \in A} \min_{e \in E} c(a, e)$$

Example

choose a_3
(maximum loss = -5)
a_1 (maximum loss = -20)
a_2 (maximum loss = -10)

c	e_1	e_2	e_3	min
a_1	40	70	-20	-20
a_2	-10	40	100	-10
a_3	20	40	-5	-5

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Remarks

- bad use of information
- no compensation between consequences in different states of Nature
- bias towards *status quo*
- only requires that X can be ordered

Example

c	e_1	e_2	e_3	$\dots$	$e_{1\,000}$
a	-100	$10\,000$	$10\,000$	$\dots$	$10\,000$
b	-99	-99	-99	$\dots$	-99

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Example

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a	-100	$10\,000$	$10\,000$	$\ldots$	$10\,000$
b	-99	-99	-99	$\ldots$	-99

Other classic criteria

- Max Max
- Hurwicz
- Min Max Regret
- Laplace

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go faster

Max Max

Idea

- optimism: base your choice on the best possible situation (Max Max)
- choose any alternative in $a \in A$ solution of:

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Example

choose a_2
(maximal gain = 100)
a_1 (maximal gain = 70)
a_3 (maximal gain = 40)

c	e_1	e_2	e_3	max
a_1	40	70	−20	70
a_2	−10	40	100	100
a_3	20	40	−5	40

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Remarks

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- only requires that X can be ordered

Hurwicz

Idea

- compromise between extreme pessimism (Max Min) and extreme optimism (Max Max)
- let $\alpha \in [0;1]$ called "coefficient of pessimism", choose any alternative $a \in A$ solution of:

$$\max_{a\in A}\left[\alpha \min_{e\in E} c(a,e) + (1-\alpha)\max_{e\in E} c(a,e)\right]$$

$\alpha = 1/2$

Choose a_2
$(90/2 = 45)$
a_1 $(50/2)$
a_3 $(35/2)$

c	e_1	e_2	e_3	min	max	$\alpha = 1/2$
a_1	40	70	-20	-20	70	25
a_2	-10	40	100	-10	100	45
a_3	20	40	-5	-5	40	$17,5$

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Remarks

- bad use of information
- compromise between bad solutions
- it must be meaningful to take linear combinations!
- how to assess the coefficient of pessimism α?

Savage (Min Max Regret)

Idea

- criterion for bureaucrats
- choose a_2 and e_1 obtains
  - best decision: a_1 (40)
  - decision taken: a_2 (-10)
  - regrets: $40 - (-10) = 50$

c	e_1	e_2	e_3
a_1	40	70	-20
a_2	-10	40	100
a_3	20	40	-5

Definition

Choose any alternative $a \in A$ solution of:

$$\min_{a \in A} \max_{e \in E} R(a, e)$$

avec

$$R(a, e) = \max_{b \in A} c(b, e) - c(a, e)$$

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R	e_1	e_2	e_3	max
a_1	0	0	120	120
a_2	50	30	0	50
a_3	20	30	105	105

Choose a_2 (max regret 50) a_1 (120), a_3 (105)

Remarks

- criterion that is different from Max Min (a_3)
- it must be meaningful to take differences!
- taking differences is an adequate way to measure regrets
- choice is set dependent. Adding new alternatives can alter choice in an unpredictable way

Example

c	e_1	e_2	e_3
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Example

c	e_1	e_2
a_1	8	0
a_2	2	4

R	e_1	e_2	max
a_1	0	4	4
a_2	6	0	6

- Choice of a_1

Example (adding a_3)

c	e_1	e_2
a_1	8	0
a_2	2	4
a_3	1	7

R	e_1	e_2	max
a_1	0	7	7
a_2	6	3	6
a_3	7	0	7

- initial choice: a_1
- choice after addition of a_3: a_2!
- risk of “manipulations”

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- initial choice: a_1
- choice after addition of a_3: a_2!
- risk of "manipulations"

Laplace

Idea

- Principle of “insufficient reason”

Definition

Choose any alternative in A solution of:

$$\max_{a \in A} \sum_{e \in E} \frac{1}{|E|} c(a,e)$$

Example

Choose a_2 (130/3)
a_1 (90/3)
a_3 (55/3)

c	e_1	e_2	e_3	
a_1	40	70	-20	90/3
a_2	-10	40	100	130/3
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Remarks

- it must be meaningful to take linear combinations!
- either you will become the King of the Belgians or not. Are these two events equally likely?
- criterion that depends on the arbitrary model for states of Nature (E can always be refined: "E and rain tomorrow" and "E and no rain tomorrow"
- Is expected gain a good criterion, even when all states are supposed equally likely?

go faster

Example

Example

c	e_1	e_2	e_3	e_4
a	2	2	0	1
b	1	1	1	1
c	0	4	0	0
d	1	3	0	0

Results

- Wald: b
- Max Max: c
- Laplace: a
- Savage: d

Example

Example

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Results

- Wald: b
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Conclusion

Classic Criteria

- none really satisfactory!
  - necessity to model likelihood (beliefs)
  - necessity to model desirability of consequences (tastes)

Central questions

- why is there no probability?
- where probabilities come from?

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Central questions

- why is there no probability?
- where probabilities come from?

Plan

Decision under risk: model

Model

- X: set of consequences
- X finite $= \{x_1, x_2, \ldots, x_n\}$
- $X \subseteq \mathbb{R}$ (e.g., money)

Simple lottery on X

- discrete r.v. on X
- $\ell = (x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$
- $p_\ell(x_i)$: probability to obtain consequence x_i with lottery ℓ

Decision under risk: model

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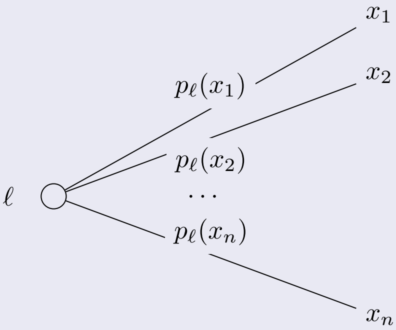

Lotteries

Set of lotteries

- simple lotteries on X
- first order lotteries on X: lotteries on simple lotteries
- second order lotteries on X: loteries on first order lotteries
- etc.

- $L(X)$: set of lotteries at all finite orders
  - $L(X)$ is always infinite

Lotteries

Remark

- $L(X)$ includes all lotteries that corresponds to the implementation of alternatives in A and many other "hypothetical" lotteries

Problem

- help someone compare lotteries in $L(X)$

Notation

- $\ell \in L(X)$: lotteries (simple or not)
- $x \in X$: consequences
- $p_\ell(x)$: probability to obtain consequence x with lottery ℓ

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Notation

- $\ell \in L(X)$: lotteries (simple or not)
- $x \in X$: consequences
- $p_\ell(x)$: probability to obtain consequence x with lottery ℓ

Lotteries

Remark

- $L(X)$ includes all lotteries that corresponds to the implementation of alternatives in A and many other "hypothetical" lotteries

Problem

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Classic Criterion

Expected Value (EV)

$$\ell \succ \ell' \Leftrightarrow \sum_{x \in X} x p_\ell(x) > \sum_{x \in X} x p_{\ell'}(x)$$

$$\ell \sim \ell' \Leftrightarrow \sum_{x \in X} x p_\ell(x) = \sum_{x \in X} x p_{\ell'}(x)$$

- $\succ$: strict preference
- $\sim$: indifference

EV

Advantages

- simple
- good use of information
- can be "decentralized"

Disadvantages

- limited to numerical consequences
- no clear rationale
- contradict observed behavior of "rational" people (diversification, insurance)

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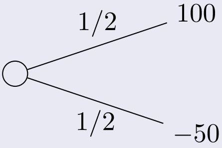
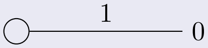


preferred to



$$E(\ell) = 25 \quad E(\ell') = 0$$

not preferred to

$$E(\ell) = 25\,000 \quad E(\ell') = 0$$

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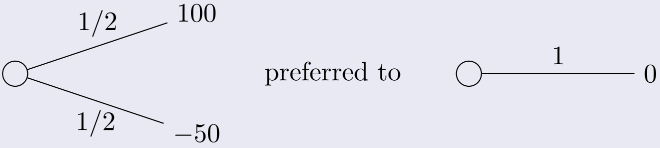

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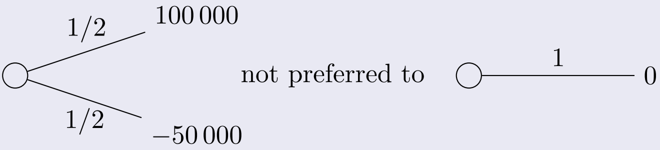

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Saint Petersburg Paradox (D. Bernoulli)

Game

- a “banker” plays with a “player”. The player must pay a fixed sum to enter the game.
- the banker tosses a coin till “Tails” obtains
- the game stops
- if “tails” obtains at the nth toss, the banker pays 2^n € to the player
- how much a rational player should be prepared to pay to enter the game?

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$$EV = 2 \times \frac{1}{2} + 2^2 \times \frac{1}{2^2} + \cdots$$

(50% chances of winning only 2 €!)

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Other classic criterion

Expected Value + Variance

- add a measure of dispersion to the measure of central tendency

Problems

- less simple
- how to deal with the two criteria (efficient solutions or synthesis?)
- is variance a good measure of risk? (inter-quartile spreads, semi-variance, etc.)

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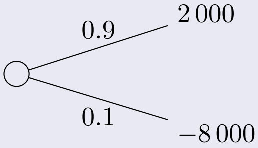

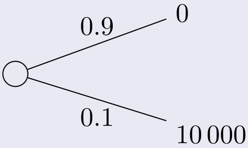

$$\begin{aligned}EV(\ell) &= 0.9 \times 2\,000 + 0.1 \times -8\,000 = 1\,000\\ &= 0.9 \times 0 + 0.1 \times 10\,000\\ Var(\ell) &= 0.9 \times (2\,000 - 1\,000)^2 + 0.1 \times (-8\,000 - 1\,000)^2\\ &= 0.9 \times 1\,000^2 + 0.1 \times 9\,000^2\\ &= 0.9 \times (0 - 1\,000)^2 + 0.1 \times (10\,000 - 1\,000)^2\end{aligned}$$

These two lotteries must be judged indifferent

Example

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These two lotteries must be judged indifferent

Limit of classic criteria

Central problem

- these criteria do not take the psychology of the individual towards risk
  - what is her wealth?
  - what is her income?
  - what is her attitude towards risk?
  - etc.

Pseudo-Solution

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- directly ask the individual about her preferences

Problems

- consistency?
- decentralized decisions?
- cognitive effort!

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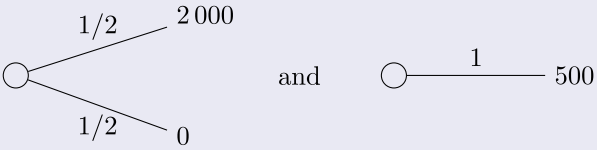

Example: choice between

- $\mathcal{N}(878.32; 72.45)$ and
- $Bi(1200; 0.75)$

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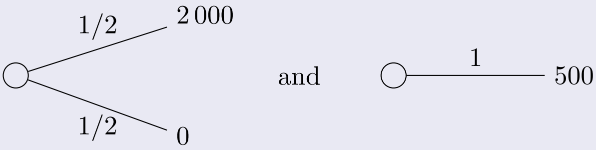

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Expected Utility Theory
J. von Neumann & O. Morgenstern (1945)

Idea

- ask the individual about simple choices
- model the behavior of the individual using a mathematical model
- use the model to process complex choices

Questions

- what model?
- what rationale?
- how to elicit the model?

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Mathematical model

Idea

- replace EV by an "Expected Utility" (EU)
- the 'utility" capture the psychology of the individual towards risk

$$\ell \succ \ell' \Leftrightarrow \sum_{x \in X} u(x) p_\ell(x) > \sum_{x \in X} u(x) p_{\ell'}(x)$$

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Utility function

- $u : X \to \mathbb{R}$
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Theoretical Analysis

How to justify the model?

- axiomatic analysis

Interpretation of axioms?

- descriptive
- normative
- prescriptive

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Axiom (A1: Ranking)

For all $\ell, \ell' \in L(X)$ at least one the following holds:

- ℓ is preferred or indifferent to ℓ ($\ell \succsim \ell'$)
- ℓ' is preferred or indifferent to ℓ ($\ell' \succsim \ell$)

Moreover, $\succsim$ is transitive:

$$\ell \succsim \ell' \text{ and } \ell' \succsim \ell'' \Rightarrow \ell \succsim \ell''$$

$\forall \ell, \ell', \ell'' \in L(X)$

Remark

- $\ell \succ \ell' \Leftrightarrow [\ell \succsim \ell'$ and Not$[\ell' \succsim \ell]]$
  - strict preference
- $\ell \sim \ell' \Leftrightarrow [\ell \succsim \ell'$ and $\ell' \succsim \ell]$
  - indifference
- A1 implies that $\sim$ and $\succ$ are transitive

Descriptive Analysis

Difficulties

- incomplete preference
- nontransitive indifference
- intransitive strict preference

Complete Preferences?

- the raison d'être of the theory is to help structure preferences!

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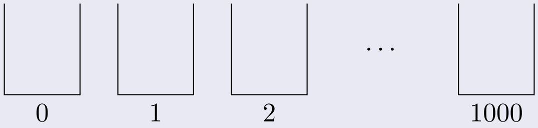
Luce (1956)

Comparaison of cups of coffee



$$0 \sim 1, 1 \sim 2, \ldots, 999 \sim 1\,000 \Rightarrow 0 \sim 1\,000$$

- imperfect senses $\Rightarrow$ nontransitive indifference

go faster

Dominance with threshold

Example

$$x \succ y \Leftrightarrow \begin{cases} x \text{ at least as good as } y \text{ on all criteria} \\ x \text{ better than } y \text{ on at least one criterion} \end{cases}$$

	g_1	g_2	g_3
a	10	10	10
b	11	11	8
c	12	9	9

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go faster

Condorcet's Paradox

Data

- Voter 1: $a \succ b \succ c$
- Voter 2: $c \succ a \succ b$
- Voter 3: $b \succ c \succ a$

$$\text{majority} : a \succ b; b \succ c, c \succ a$$

Threshold effects

Example

1	Car	15 000 €
2	Car + PE1	15 500 €
3	Car + PE1 + PE2	16 000 €
4	Car + PE1 + PE2 + PE3	16 500 €
⋮		
n	Car + ...	18 000 €

Preference of a "naïve" consumer

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Prescriptive Approach

- effectiveness
  - it is simple to help someone choose on the basis of complete and transitive preferences

Normative Approach

- money pump argument
  - exchanges starting with c

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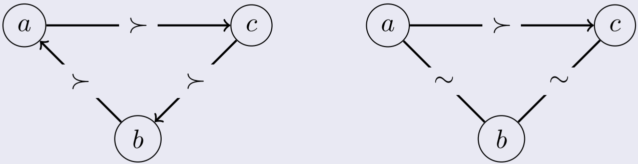

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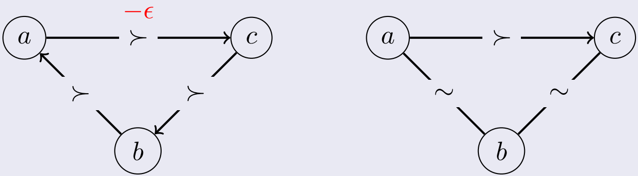

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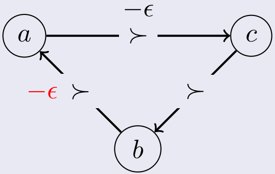

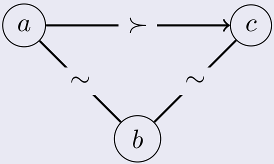

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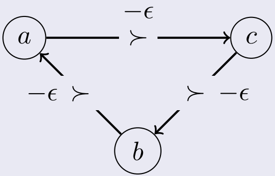

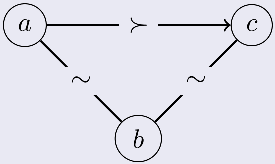

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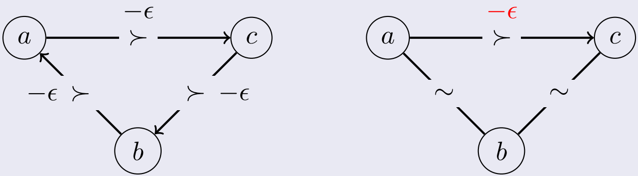

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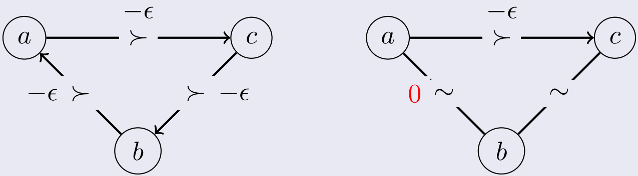

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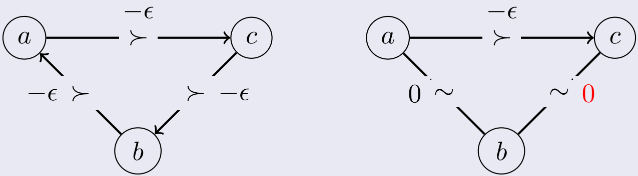
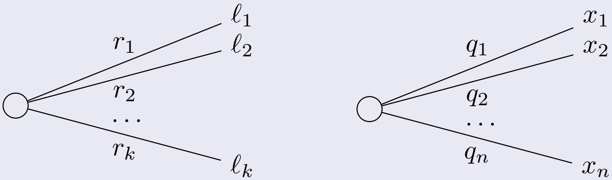


Axioms

Axiom (A2 Reduction)

ℓ_j: first order lotteries



with $q_i = \sum_{j=1}^{k} r_j p_{\ell_j}(x_i)$

$\Rightarrow$ Indifference

Interpretation

- "games are played seriously"

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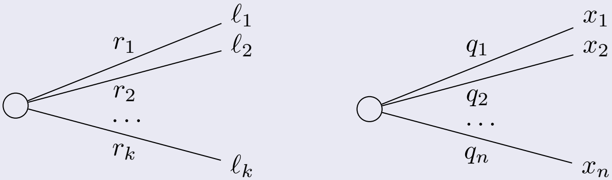

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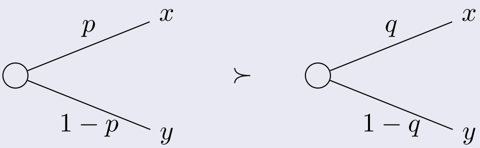

iff $p > q$ $(\forall x, y \in X)$

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- greediness
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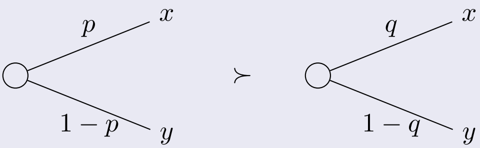

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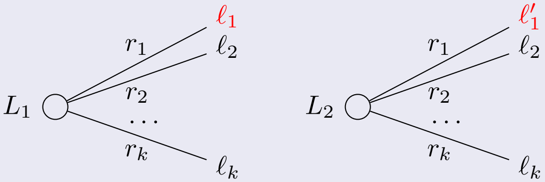

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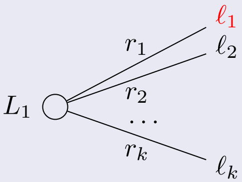

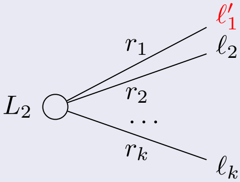

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Remark

- A3 implies that this probability is unique

Interpretation

- you are not naïve with probabilities (continuum between certainty and risk)

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- x: win 2€
- y: win 1€
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$$(x, 1) \succ (y, 1) \succ (z, 1)$$

Problem

- is there a probability $p \in]0; 1[$ such that:

$$y \sim (x, p; z, (1 - p))$$

- $p = 1 - 10^{-100}$?

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Consequences of axioms

Theorem (Representation)

Let $\succsim$ be a preference relation on $L(X)$.
This relation satisfies A1-A5
iff
there is a function $u : X \to \mathbb{R}$ such that:

$$\ell \succsim \ell' \Leftrightarrow \sum_{x \in X} u(x) p_\ell(x) \geq \sum_{x \in X} u(x) p_{\ell'}(x) \qquad \text{(vNM)}$$

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- necessity is obvious
- u is linked to $\succsim$ and, hence, to the individual

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skip proof

Proof

5 steps

- constructive!

Finite case: $X = \{x_1, x_2, \ldots, x_n\}$

Consider a lottery $\ell \in L(X)$

1)

Using A1, (ranking), A2 (reduction) and A4 (independence), you can always find a simple lottery such that: $\ell \sim (x_1, p_\ell(x_1); x_2, p_\ell(x_2); \ldots ; x_n, p_\ell(x_1))$

Suppose wlog that:

$$(x_n, 1) \succ (x_{n-1}, 1) \succ \cdots \succ (x_1, 1)$$

Proof

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2)
A5 (continuity): since

$$(x_n, 1) \succ (x_{n-1}, 1) \succ \cdots \succ (x_1, 1)$$

there is $u_i \in \,]0;1[$ such that

$$(x_i, 1) \sim [x_n, u_i; x_1; (1 - u_i)]$$

Let:

$$u_n = 1, u_1 = 0$$

3)
Using A4 (independence), A1 (ranking) and A2 (reduction), we know that:

$$\ell \sim (x_1, p_\ell(x_1); x_2, p_\ell(x_2); \ldots; x_{n-1}, p_\ell(x_{n-1}); x_n, p_\ell(x_n))$$

$$\ell \sim [x_1, (1 - K_\ell); x_2, 0; \ldots; x_{n-1}, 0; x_n, K_\ell]$$

with

$$K_\ell = \sum_{i=1}^{n} p_\ell(x_i) u_i$$

4)
Use steps 1) to 3) to transform a lottery
$\ell' \sim (x_1, p_{\ell'}(x_1); x_2, p_{\ell'}(x_2); \ldots; x_{n-1}, p_{\ell'}(x_{n-1}); x_n, p_{\ell'}(x_n))$
We have:

$$\ell' \sim [x_1, (1 - K_{\ell'}); x_2, 0; \ldots; x_{n-1}, 0; x_n, K_{\ell'}]$$

with

$$K_{\ell'} = \sum_{i=1}^{n} p_{\ell'}(x_i) u_i$$

5)
Using A1 (Ranking) and A3 (Monotonicity) we know that:

$$\ell \succ \ell' \Leftrightarrow$$

$$(x_1, p_\ell(x_1); x_2, p_\ell(x_2); \ldots; x_n, p_\ell(x_n)) \succ$$

$$(x_1, p_{\ell'}(x_1); x_2, p_{\ell'}(x_2); \ldots; x_n, p_{\ell'}(x_n)) \Leftrightarrow$$

$$[x_1, (1 - K_\ell); x_2, 0; \ldots; x_n, K_\ell] \succ [x_1, (1 - K_{\ell'}); x_2, 0; \ldots; x_n, K_{\ell'}]$$

$$\Leftrightarrow$$

$$K_\ell > K_{\ell'} \Leftrightarrow$$

$$\sum_{i=1}^{n} p_\ell(x_i) u_i > \sum_{i=1}^{n} p_{\ell'}(x_i) u_i$$

and define u letting:

$$u(x_i) = u_i$$

□

Consequences of axioms

Theorem (Uniqueness)

If there are two functions u and v such that (vNM) holds then there are $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that:

$$v(x) = \alpha u(x) + \beta$$

$\forall x \in X$

Interpretation

preferences can be measured as temperature

Proof

obvious: if u and v are not linked by a positive affine transformation, you can always find two loteries that will have different expected utilities □

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Assessment of a utility function

Hypotheses

- $X = \mathbb{R}$ (money)
- let $u(0) = 0$ and $u(10\,000) = 1$

Assessment

$$1 \times u(x) = u(x) = 1/2 \times u(10\,000) + 1/2 \times u(0) = 1/2$$

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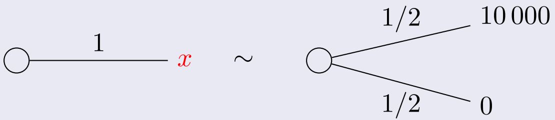

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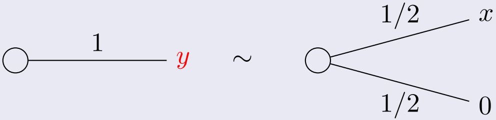
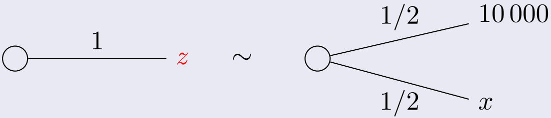


$u(y) = 1/2 \times u(x) + 1/2 \times u(0) = 1/4$

Assessment of a utility function

Assessment

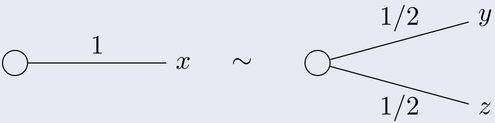


$$u(z) = 1/2 \times u(10000) + 1/2 \times u(x) = 3/4$$

Assessment of a utility function

Control question

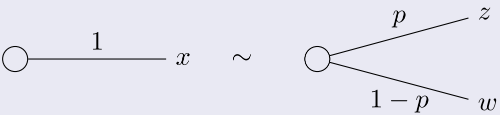
- we must have:



- if not: go back and check

Assessment of a utility function

General case

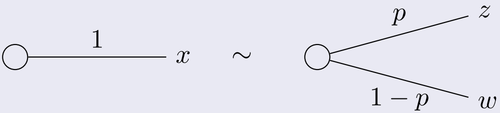


$u(x) = pu(z) + (1-p)u(w)$

- 4 unknowns
- fix the value of 3 of them and find indifference on the 4th

Assessment of a utility function

General case



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- 4 unknowns
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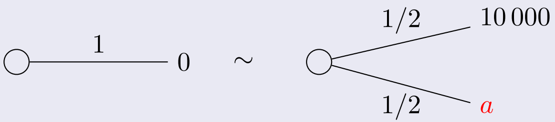
go faster

Assessment of a utility function

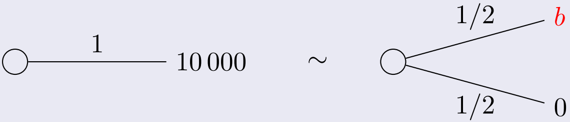
Going down



$$1 \times u(0) = 1/2 \times u(10\,000) + 1/2 \times u(a) \Rightarrow u(a) = -1$$

Assessment of a utility function

Going up



$1 \times u(10\,000) = 1/2 \times u(b) + 1/2 \times u(0) \Rightarrow u(b) = 2$

Assessment of a utility function

Remarks

- use of simple probabilities: 1/2, 1/3, 1/4
- bracketing
- checks are necessary

Risk aversion

Context

- let $X = \mathbb{R}$ (money)
- suppose that the DM satisfies A1-A5
- it is not restrictive to suppose that u is increasing!
- P: wealth of the DM

Certainty equivalent of a lottery ℓ: $\hat{x}(\ell)$

- amount of money that the DM finds equivalent to owning the lottery ℓ (minimal selling price of lottery ℓ)

$$E(u(P + \hat{x}(\ell))) = u(P + \hat{x}(\ell)) = E(u(P + \ell))$$

$$\Rightarrow$$

$$\hat{x}(\ell) = u^{-1}[E(u(P + \ell))] - P$$

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Risk premium $\pi(\ell)$ for lottery ℓ

Risk premium

$$\pi(\ell) = E(\ell) - \hat{x}(\ell)$$

Risk aversion

A DM is

- risk averse if $\hat{x}(\ell) < E(\ell)$
- risk prone if $\hat{x}(\ell) > E(\ell)$
- risk neutral if $\hat{x}(\ell) = E(\ell)$ (EV)

pour toute loterie $\ell \in L(X)$

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Risk aversion

Theorem (Arrow-Pratt)

A DM is risk averse iff her utility function is concave

Proof

Risk aversion ($\hat{x}(\ell) < E(\ell)$)

$$E[u(P + E(\ell))] = u(P + E(\ell)) > E(u(P + \ell)) = u(P + \hat{x}(\ell))$$

$$u(P + [\alpha x + (1 - \alpha)y]) = u(\alpha(x + P) + (1 - \alpha)(y + P)) > E(u(P + \ell)) = \alpha u(x + P) + (1 - \alpha)u(y + P)$$

$z = x + P$, $w = y + P$

$u(\alpha z + (1 - \alpha)w) > \alpha u(z) + (1 - \alpha)u(w)$ □

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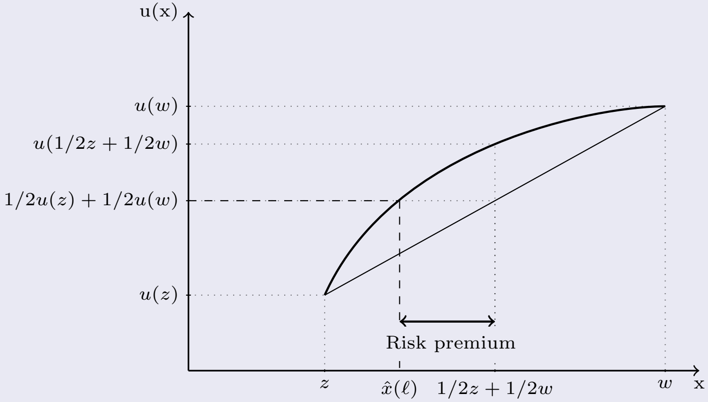
$u(\alpha z + (1 - \alpha)w) > \alpha u(z) + (1 - \alpha)u(w)$ $\square$


u(x)
$u(w)$
$u(1/2z + 1/2w)$
$1/2u(z) + 1/2u(w)$
$u(z)$
Risk premium
z
$\hat{x}(\ell)$
$1/2z + 1/2w$
w
x

Plan

Decision under uncertainty

Subjectivist school

- de Finetti, Savage, Aunscombe & Aumann
- use results in the context of risk in the context of uncertainty

Idea

Minimal greediness

- if the first lottery is preferred (not preferred) to the second lottery we have $P(E) > p$ (resp. $P(E) < p$)
- the subjective probability of the event is the value p for which these two lotteries are indifferent

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- under uncertainty, use subjective probabilities
- back to the case of decision under risk

Questions

- are subjective probabilities probabilities?
- are subjective probabilities “true” probabilities?
- rationale?

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go faster

Reminder

Random experiment

- experiment for which it is impossible to predict the result before you run it

Examples

- toss of a coin
- price of $/€ in one year
- next month sales

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Definitions

- S: set of elementary events = sets of all possible results of the experiment
  - toss of a coin: $S = \{H, T\}$
  - rolling a dice: $S = \{1, 2, 3, 4, 5, 6\}$
  - price of \$/€ in one year: $S = \mathbb{R}_+$
- $\mathcal{S}$: sets of events based on S (allowing to speak of the union, intersection, of elementary events)

Properties

- $S \in \mathcal{S}$
- $A \in \mathcal{S} \Rightarrow S \setminus A \in \mathcal{S}$
- $A, B \in \mathcal{S} \Rightarrow A \cup B \in \mathcal{S}$

(algebra, σ-algebra)

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(algebra, σ-algebra)

Definition

A probability P is a mapping from $\mathcal{S}$ to $\mathbb{R}$, associating to each event A its probability $P(A)$ and such that:

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $A \cap B = \varnothing \Rightarrow P(A \cup B) = P(A) + P(B)$

(Kolmogorov's axioms)

go faster

Reminder

Definition

The probability of "A given B", denoted by $P(A/B)$, is defined by:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Property

$$P(A/B) = \frac{P(B/A)P(A)}{P(B)}$$

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Total probabilities

If $A_1, A_2, ..., A_n$ partition S then:

$$P(A) = \sum_{i=1}^{n} P(A/A_i)P(A_i)$$

Bayes' formula

$$P(A/B) = \frac{P(B/A)P(A)}{\sum_{i=1}^{n} P(B/A_i)P(A_i)}$$

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Probabilities?

Source of probabilities?

- two main schools
  - classic school
  - subjectivist school

Classic school

Sources of probabilities

- logical source
- frequentist source

Logical source

Principle of “insufficient reason”

- $P(\text{Tails}) = P(\text{Heads}) = 1/2$
- $P(E) = \text{Number of favorable cases}/\text{Number of cases}$

Problems

- restricted to “perfect objects”
- problem of proof: what is a “perfect object”?
- limited practical impact (ex.: price of $/€ in one year?)

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Frequentist source

Law of large numbers

- n identical and independent repetitions of event E of probability p:

$$\lim_{n\to\infty} P\left(\left|\frac{n(E)}{n} - p\right| > \varepsilon\right) = 0$$

- "if you do not p, experiment"

Problems

- events that cannot be "repeated"
- proof problem (past = future)
- different results if "identical"?
- rôle of information unclear

Exemple: thumbtack

- 10 000 € if Tails
- −5 000 € if Heads

Question

would you accept to perform 1 000 tosses of the thumbtack before taking a decision?

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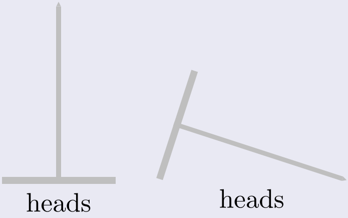

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Subjectivist school

Facts

- 2 different and equally rational persons cas have different probability judgements on the same event
- experiments transform probability judgements: thumbtack
- language has a great variety of expression to speak of likelihood

Definition: subjective probability

- a probability is a subjective measure of the likelihood of an event

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Problems

- why probabilities? (Kolmogorov's axioms)
- how to assess probabilities?

Idea

- comparing "uncertain" lotteries with "risky lotteries" (with probabilities coming from consensual random experiments; coins, cards, dices, urns, etc.)

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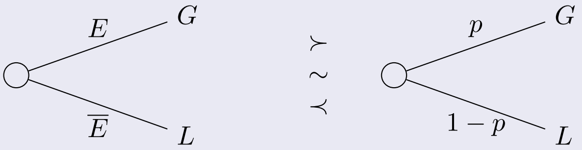

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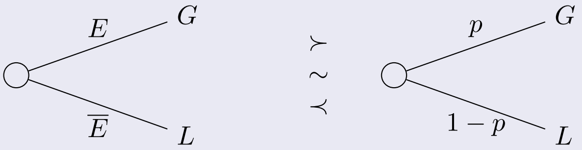

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Problems

- are these numbers probabilities?
- are these numbers true probabilities?
- rationale?
- use?

Results

Idea

- add to $L(X)$ "uncertain lotteries"
- impose (modified) axioms A1-A5 to this new set of lotteries

Theorem (Savage, Aunscombe & Aumann)

There are a function $u : X \to \mathbb{R}$ and a Probability measure P on S such that the comparison of two lotteries (risky or uncertain) is done comparing their Expected Utilities (p for risky lotteries, P for risky lotteries)

The probability measure is unique. The utility function is unique up to a positive affine transformation.

- tastes: u (subjective)
- beliefs: P (subjective)

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The probability measure is unique. The utility function is unique up to a positive affine transformation.

- tastes: u (subjective)
- beliefs: P (subjective)

Consequences

- same analysis as in the risky case replacing (whenever needed) "probabilities" by "subjective probabilities"
- exchangeable events and subjective vs objective probabilities (de Finetti)

Idea of proof

Example ($A \cap B = \varnothing$)

L_1	F	P
A	G	L
B	L	G
A or B	L	L

L_2	F	P
A	G	L
B	G	L
A or B	L	L

A2 $\Rightarrow$ [G if Tails; L if Heads] $\sim$ [L if Tails; G if Heads]
A4 $\Rightarrow$ $L_1 \sim L_2$

Idea of proof

Example ($A \cap B = \varnothing$)

L_1	F	P
A	G	L
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$\overline{A \text{ or } B}$	L	L

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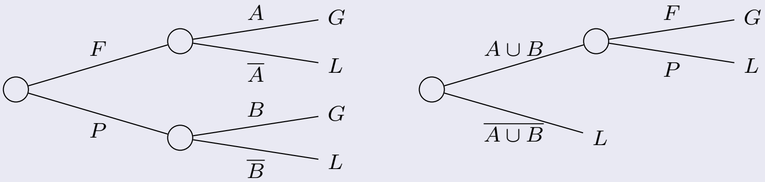
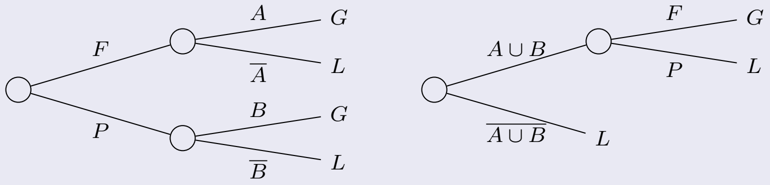


Consequences

- $A4 \Rightarrow L_1 \sim L_2$
- Let $u(G) = 1$ and $u(P) = 0$
- We have:

$$
\begin{aligned}
E[u(L1)] =& 0.5[P(A)u(G) + (1 - P(A))u(L)] + \\
& 0.5[P(B)u(G) + (1 - P(B))u(L)] \\
& 0.5[P(A) + P(B)] \\
E[u(L2)] =& 0.5P(A \cup B)
\end{aligned}
\quad \Rightarrow \quad P(A) + P(B) = P(A \cup B)
$$

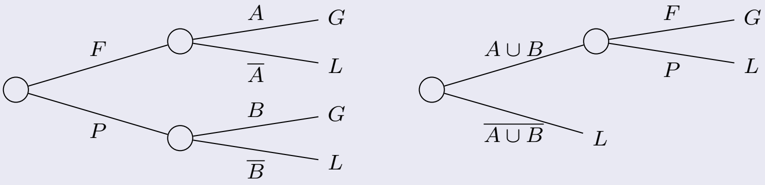


Consequences

- $A4 \Rightarrow L_1 \sim L_2$
- Let $u(G) = 1$ and $u(P) = 0$
- We have:

$$
\begin{aligned}
E[u(L1)] =& 0.5[P(A)u(G) + (1 - P(A))u(L)] + \\
& 0.5[P(B)u(G) + (1 - P(B))u(L)] \\
& 0.5[P(A) + P(B)] \\
E[u(L2)] =& 0.5P(A \cup B)
\end{aligned}
$$

$$\Rightarrow P(A) + P(B) = P(A \cup B)$$



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$$\Rightarrow$$
$$P(A) + P(B) = P(A \cup B)$$

Bayesian Decision Theory

creativity:	A
objectives:	X
beliefs:	P
tastes:	u
evaluation:	$(S)EU$

Applications

- risk aversion
- stochastic dominance
- economics of finance and insurance
- value of information
- Bayesian statistics

- all this must be skipped today :-(

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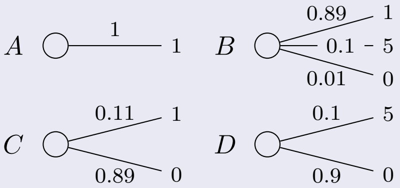
Plan

Allais' paradox (10^6 €)

Experiment



Modal Result

- $A \succ B$ and $D \succ C$

Interpretation

- $A \succ B \Rightarrow$
  $u(1) > 0.89u(1) + 0.1u(5) + 0.01u(0)$
  $\Rightarrow u(1) > 10/11$
- $D \succ C \Rightarrow$
  $0.1u(5) + 0.9u(0) > 0.11u(1) + 0.89u(0)$
  $\Rightarrow u(1) < 10/11$

Allais' paradox (10^6 €)

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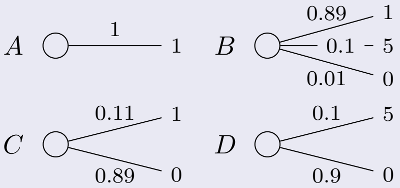
Experiment



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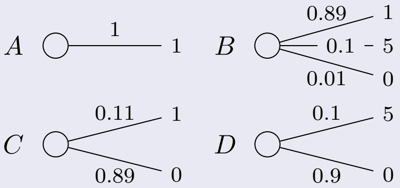

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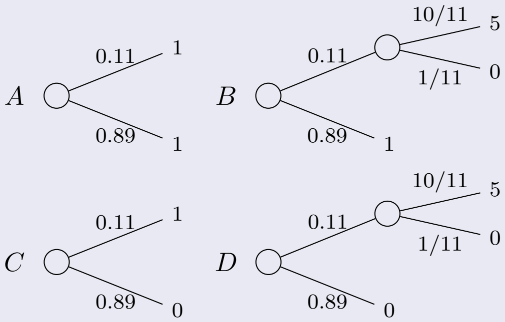
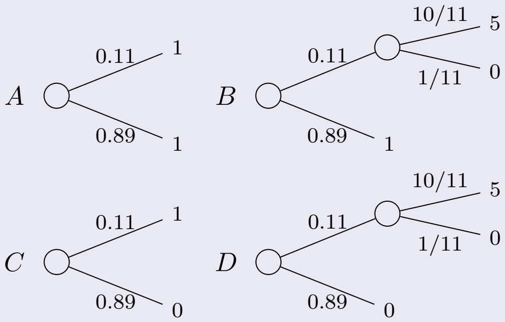
Allais' paradox

Reformulation



Allais' paradox

Reformulation



skip examples

Allais's paradox

Reformulation

- urn with 89 balls R and 11 balls N

	R	N
X	Q	1
Y	Q	5 with 10/11 0 with 1/11

Should the choice between X and Y depend upon Q?

- $Q = 0$: $X = C$ and $Y = D$
- $Q = 1$: $X = A$ and $Y = B$

Violation of independence

Normative analysis

- $(x;1) \succ (y;1)$ and $(y,p;z,(1-p)) \succ (x,p;z,(1-p))$
- $(x;1) \succ (y;1) \Rightarrow (x-\varepsilon;1) \succ (y;1)$
- $(y,p;z,(1-p)) \succ (x,p;z,(1-p)) \Rightarrow$
  $(y,p;z-\varepsilon,(1-p)) \succ (x,p;z,(1-p))$

Exchanges starting with $(x,p;z,(1-p))$

- $(x,p;z,(1-p))$ exchanged for $(y,p;z-\varepsilon,(1-p))$
  - if E does not obtain: gain $= (z-\varepsilon)$
  - if E obtains: gain $= y$ and y is exchanged for $(x-\varepsilon)$
- you had $(x,p;z,(1-p))$ you have $(x-\varepsilon,p;z-\varepsilon,(1-p))$!

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Rational violation of independence

Mom's decision

- two kids: A and B
- only one gift to be given to either A or B

Mom's preferences

- $(A, 1) \sim (B, 1)$
- $(B, 1/2; A, 1/2) \succ (A, 1/2; A, 1/2)$

$\Rightarrow$ rational violation of independence

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$\Rightarrow$ rational violation of independence

EURDP

Definition

- distortion of probabilities according to their rank
- ϕ increasing bijection on $[0,1]$

$$(x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$$

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$$

where $(\cdot)$ is a permutation on $\{1, 2, \ldots, n\}$

$$(x_{(1)}, p_{(1)}; x_{(2)}, p_{(2)}; \ldots; x_{(n)}, p_{(n)})$$

EURDP

Computation

- ϕ increasing bijection on $[0, 1]$

$$
\begin{aligned}
EURDP = {} & u(x_{(1)})\phi(p_{(1)} + p_{(2)} + \dots p_{(n)}) + \\
& (u(x_{(2)}) - u(x_{(1)}))\phi(p_{(2)} + \dots p_{(n)}) + \\
& (u(x_{(3)}) - u(x_{(2)}))\phi(p_{(3)} + \dots p_{(n)}) + \\
& (u(x_{(4)}) - u(x_{(3)}))\phi(p_{(4)} + \dots p_{(n)}) + \\
& \dots \\
& (u(x_{(n)}) - u(x_{(n-1)}))\phi(p_{(n)})
\end{aligned}
$$

Motivation

- distorting cumulative probabilities is inevitable if inconsistencies are to be avoided

EURDP

Computation

- ϕ increasing bijection on $[0, 1]$

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Motivation

- distorting cumulative probabilities is inevitable if inconsistencies are to be avoided

Allais' paradox and EURDP

Allais

- $A : (1, 1)$ preferred to $B : (0, 0.01; 1, 0.89; 5, 0.1)$
- $D : (0, 0.9; 5, 0.1)$ preferred to $C : (0, 0.89; 1, 0.11)$

$$EURDP(A) = u(1)$$
$$EURDP(B) = u(0) + (u(1) - u(0))\phi(0.99) + (u(5) - u(1))\phi(0.1)$$
$$EURDP(C) = u(0) + (u(1) - u(0))\phi(0.11)$$
$$EURDP(D) = u(0) + (u(5) - u(0))\phi(0.1)$$

Allais' paradox and EURDP

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$$EURDP(D) = u(0) + (u(5) - u(0))\phi(0.1)$$

- $u(5) = 1$, $u(0) = 0$, $\phi(x) = x^2$

$$EURDP(A) = u(1) \quad EURDP(B) = 0.01 - 0.9701u(1)$$
$$EURDP(C) = 0.11^2 u(1) \quad EURDP(D) = 0.1^2$$
$$u(1) > 0.01 - 0.9701u(1) \Rightarrow u(1) > 0.33$$
$$0.1^2 > 0.11^2 u(1) \Rightarrow u(1) < 0.826$$

Interpretation

EU & EURDP

- $\phi(x) = x^2$
- $(x_1, p; x_2, (1-p))$ avec $x_1 < x_2$
- $EU = pu(x_1) + (1-p)u(x_2)$

$$\begin{aligned} EURDP &= u(x_1) + (u(x_2) - u(x_1))\phi(1-p) \\ &= (1 - \phi(1-p))u(x_1) + \phi(1-p)u(x_2) \end{aligned}$$

- $\phi(x) = x^2$ convex $\Rightarrow$
- $\phi(1-p) < (1-p)$ and $(1 - \phi(1-p)) > p$
- bad consequences are overweighed
- good consequences are underweighed

Pessimism

Interpretation

Interpretation: EURDP

- u captures attitude towards consequences
- ϕ captures attitude towards risk
- with EU u plays both rôles

EURDP

- huge literature on the axiomatic foundations and experimental validity
- distorting probabilities
- envisage other models of likelihood

Interpretation

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Belief functions

Example

- you know that a proportion α_k of vehicles are of type k
- you know that a vehicle of type k has a length between ℓ_k^- and ℓ_k^+
- an available slot in which a vehicle of length L can park

Probability that a car can park?

- between

$$\sum_{k:\ell_k^+ \leq L} \alpha_k$$

and

$$\sum_{k:\ell_k^- \leq L} \alpha_k$$

Belief functions

$$\varphi([\ell_k^-, \ell_k^+]) = \alpha_k$$

- inferior probability

$$p^-([0,L]) = \sum_{[\ell_k^-,\ell_k^+]\subseteq[0,L]} \varphi([\ell_k^-, \ell_k^+])$$

- superior probability

$$p^+([0,L]) = \sum_{[\ell_k^-,\ell_k^+]\cap[0,L]\neq\varnothing} \varphi([\ell_k^-, \ell_k^+])$$

- inferior probability and Möbius masses

$$p^-(A) = \sum_{B\subseteq A} \varphi(B) \Leftrightarrow \varphi(A) = \sum_{B\subseteq A} (-1)^{|A\setminus B|} p^-(B)$$

Belief functions

$$p^-(A) = \sum_{B \subseteq A} \varphi(B) \quad p^+(A) = \sum_{B \cap A \neq \varnothing} \varphi(B)$$

- knowing p^- is equivalent to knowing p^+

$$p^+(A) = 1 - p^-(\overline{A})$$

- knowing p^- is equivalent to knowing φ

$$\varphi(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} p^-(B)$$

Belief functions

Definition

- p^-: Belief function (Bel)
- p^+: Plausibility function (Pl)
- φ: Möbius masses

- $p^-(\varnothing) = 0$, $p^-(S) = 1$
- $B \subseteq A \Rightarrow p^-(B) \leq p^-(A)$
- $p^-(A \cup B) \geq p^-(A) + p^-(B) - p^-(A \cap B)$
- $p^-(A \cup B \cup C) \geq$ $p^-(A)+p^-(B)+p^-(C)-p^-(A\cap B)-p^-(A\cap C)-p^-(B\cap C)+p^-(A\cap B\cap C)$
- p^- monotone at all orders

- $\varphi(\varnothing) = 0$, $\varphi(B) \geq 0$
- $\sum_{B \subseteq S} \varphi(B) = 1$

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- $\varphi(\varnothing) = 0$, $\varphi(B) \geq 0$
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Belief functions

Probabilities

- $\varphi(A) > 0 \Rightarrow |A| = 1$: masses only on singletons
- $p^- = p^+$: probabilities

Possibilities

- $\varphi(A) > 0$ and $\varphi(B) > 0 \Rightarrow A \subseteq B$ or $B \subseteq A$: embedded masses
- p^-: necessity measure
- p^+: possibility measure

$$Nec(A \cup B) = \min(Nec(A), Nec(B))$$
$$Pos(A \cup B) = \max(Pos(A), Pos(B))$$

Belief functions

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Ellsberg's Paradox

Experiment

- 90 balls in an urn: 30 are R and 60 are B or Y
- $a1$: 100 if R or $a2$: 100 if B
- $a3$: 100 if R or Y or $a4$: 100 if B or Y
- model answer: $a1 \succ a2$ and $a4 \succ a3$
- incompatible with SEU!

$$E[u(a1)] = P(R)u(100) \quad E[u(a2)] = P(B)u(100)$$

$$a1 \succ a2 \Rightarrow P(R) > P(B)$$

$$E[u(a3)] = P(R)u(100) + P(Y)u(100) \quad E[u(a4)] = P(B)u(100) + P(Y)u(100)$$

$$a4 \succ a3 \Rightarrow P(B) > P(R)$$

Aversion to ambiguity

Ellsberg's Paradox

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Aversion to ambiguity

Choquet Expected Utility (CEU)

Capacity

$$C : 2^S \to [0,1]$$

$$A \subseteq B \Rightarrow C(A) \leq C(B)$$

$$C(\varnothing) = 0, C(S) = 1$$

Capacity: belief function that is not necessarily monotone

CEU

$(x_{(1)}, E_{(1)}; x_{(2)}, E_{(2)}; \ldots; x_{(n)}, E_{(n)})$ avec $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$

$$\begin{aligned} CEU = & \; u(x_{(1)})C(E_{(1)} \cup E_{(2)} \cup \cdots \cup E_{(n)}) + \\ & (u(x_{(2)}) - u(x_{(1)}))C(E_{(2)} \cup E_{(3)} \cup \cdots \cup E_{(n)}) + \\ & (u(x_{(3)}) - u(x_{(2)}))C(E_{(3)} \cup E_{(4)} \cup \cdots \cup E_{(n)}) + \\ & \vdots + \\ & (u(x_{(n)}) - u(x_{(n-1)}))C(E_{(n)}) \end{aligned}$$

CEU and Ellsberg

- 90 balls in an urn: 30 are R and 60 are B or Y
- $a1$: 100 if R or $a2$: 100 if B
- $a3$: 100 if R or Y or $a4$: 100 if B or Y
- $a1 \succ a2$ and $a4 \succ a3$

$$a1 \succ a2 \Rightarrow C(R) > C(B)$$
$$a4 \succ a3 \Rightarrow C(B \cup Y) > C(R \cup B)$$

Particular cases

- $A \cap B = \varnothing \Rightarrow C(A) + C(B) = C(A \cup B)$: Probability
- $C(A \cap B) = \max(C(A), C(B))$: Possibility
- $C(A \cap B) = f(C(A), C(B))$: Decomposable Capacity
- C monotone: Belief functions

Capacity

Möbius masses

$$\varphi(A) = \sum_{B \subseteq A} -1^{|A \setminus B|} C(B)$$

Ellsberg

	$\varnothing$	R	B	Y	$R \cup B$	$R \cup Y$	$B \cup Y$	S
C	0	1/3	0	0	1/3	1/3	2/3	1
φ	0	1/3	0	0	0	0	2/3	0

Extensions

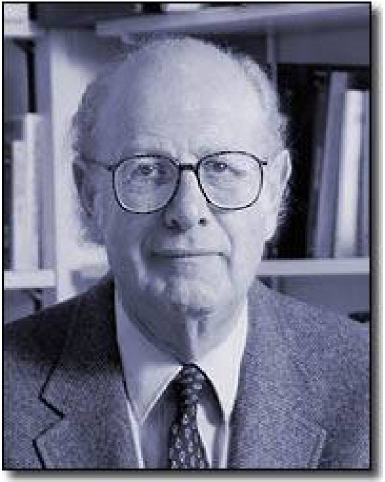
Cumulative prospect theory

- rank dependence
- sign dependence: reference effects

Bruno de Finetti

Leonard J. Savage

Howard Raiffa



Daniel Kahneman

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C'EST TOUT POUR AUJOURD'HUI.