Math-H-405 - Decision engineering

Solutions of Session 5: Decision making under risk and uncertainty

REMINDER:
$A_1, A_2, \ldots, A_n$ are a partition of Ω if $A_i \cap A_j = \phi \ \forall i \neq j$ and if $\bigcup_{k=1}^{n} A_k = \Omega$.
Total probabilities law: if $A_1, A_2, \ldots, A_n$ are partition of Ω then

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

Bayes formula:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}.$$

Exercise 1

1. (a) If we want to maximize the maximum, the solution is to maintain (see Table 1).
   (b) If we want to maximize the minimum, the solution is to expand (see Table 1).

Decision	**Good Foreign**	**Poor Foreign**		
			MAX	**MIN**
E	800 000	5000 000	800 000	500 000
M	1 300 000	-150 000	1 300 000	-150 000
S	320 000	320 000	320 000	320 000

Table 1: MAX MAX criterion and MAX MIN criterion

   (c) If we want to minimize the maximum regret, the solution is to expand (see table 2).

$R(a,e)$	**Good Foreign**	**Poor Foreign**	**MAX**
E	500 000	0	500 000
M	0	650 000	650 000
S	980 000	180 000	980 000

Table 2: Table of regret $R(a,e)$ for each $a \in A$ and $e \in E$.

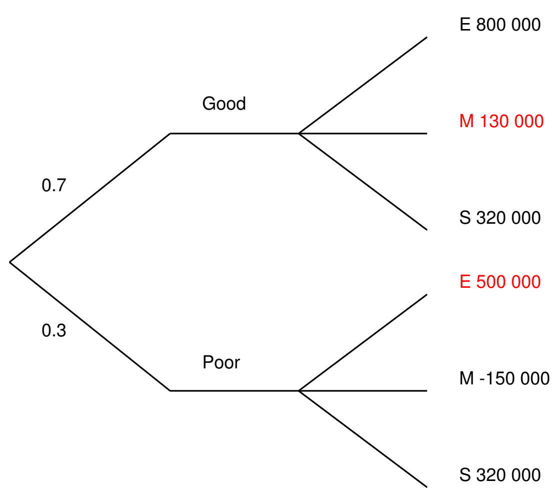
(d) If we apply the Hurwicz criterion with $\alpha = 0.7$, the solution is to expand:
$E : 800\ 000 * 0.3 + 500\ 000 * 0.7 = 590\ 000$
$M : 1\ 300\ 000 * 0.3 - 150\ 000 * 0.7 = 285\ 000$
$S : 320\ 000 * 0.3 + 320\ 000 * 0.7 = 320\ 000$

(e) If we apply the Laplace criterion, the solution is to expand:
$E : 800\ 000 * 0.5 + 500\ 000 * 0.5 = 650\ 000$
$M : 1\ 300\ 000 * 0.5 - 150\ 000 * 0.5 = 575\ 000$
$S : 320\ 000 * 0.5 + 320\ 000 * 0.5 = 320\ 000$

2. If we apply the mathematical expectation criterion, the solution is to maintain:
$E[E] = 800\ 000 * 0.7 + 500\ 000 * 0.3 = 710\ 000$
$E[M] = 1\ 300\ 000 * 0.7 - 150\ 000 * 0.3 = 865\ 000$
$E[S] = 320\ 000 * 0.7 + 320\ 000 * 0.3 = 320\ 000$
If we apply the "expected opportunity loss" criterion (regret expectation), the solution is to maintain:
$E[E] = 500\ 000 * 0.7 + 0 * 0.3 = 350\ 000$
$E[M] = 0 * 0.7 + 650\ 000 * 0.3 = 195\ 000$
$E[S] = 980\ 000 * 0.7 + 180\ 000 * 0.3 = 740\ 000$
We can note that the strategy "to sell" is actually dominated by the two others, we could thus have suppressed it.

3. $E[Gain] = 0.7 * 1\ 300\ 000 + 0.3 * 500\ 000 = 1\ 060\ 000$
It is the price we will be ready to pay for a perfect information.

4. Decision tree:



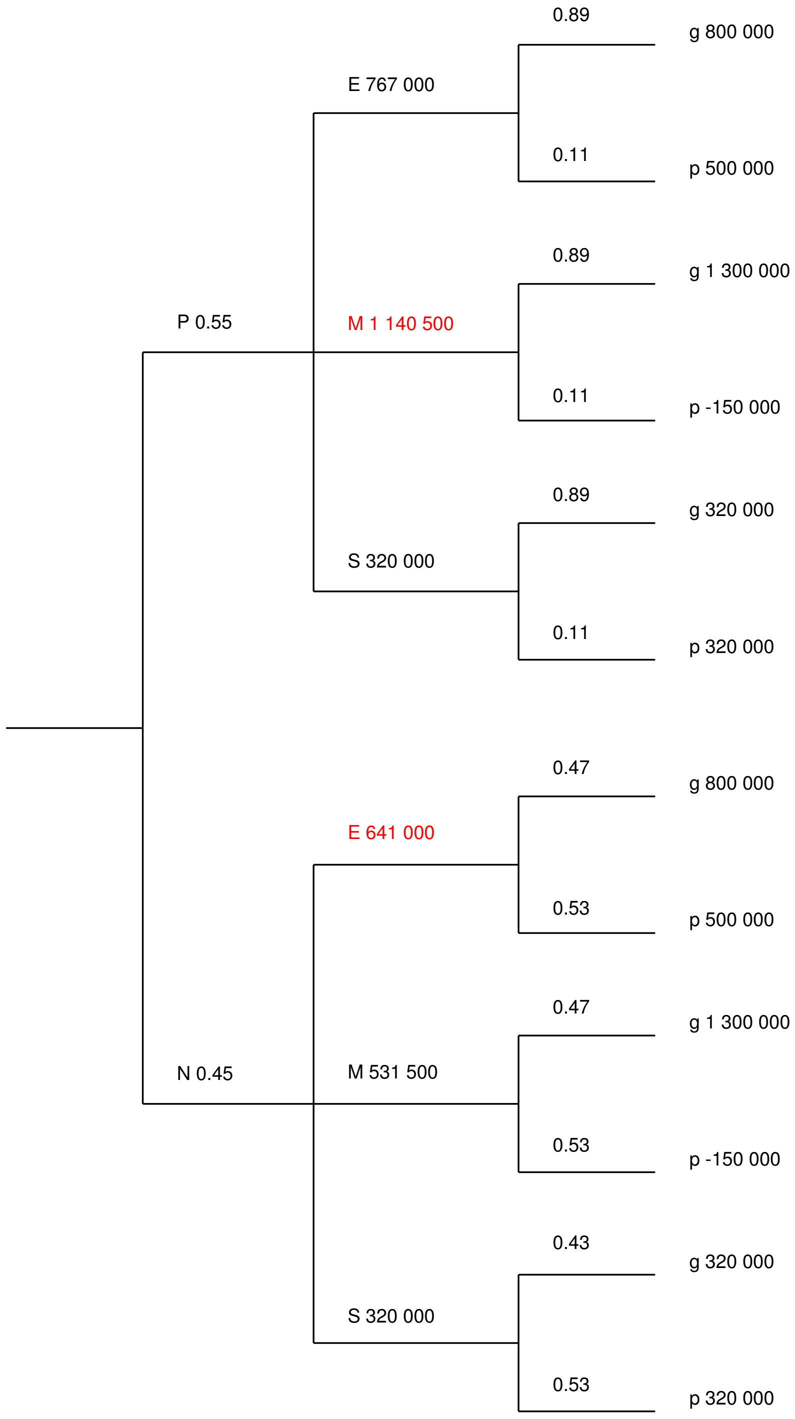
5. Let us note p and g respectively the events "unfavourable situations" and "favourable situations" and let us note N and P respectively the events "obtain a negative report" and "obtain a positive report". We know that $P(p) = 0,3$ and $P(g) = 0,7$. We also know that $P(P|g) = 0,7$, $P(N|g) = 0,3$, $P(P|p) = 0,2$ and $P(N|p) = 0,8$. We can thus compute $P(P)$ and $P(N)$ using the total probabilities law:

$$\begin{aligned} P(P) &= P(P|g)P(g) + P(P|p)P(p) = 0,7 * 0,7 + 0,2 * 0.3 = 0,55 \\ P(N) &= 1 - P(P) = 0.45 \end{aligned}$$

We can also compute $P(g|P)$, $P(g|N)$, $P(p|P)$ and $P(p|N)$ using the Bayes formula:

$$\begin{aligned} P(g|P) &= \frac{P(P|g)P(g)}{P(P)} = \frac{0,7 * 0,7}{0,55} = 0.89 \\ P(p|P) &= 1 - P(g|p) = 0.11 \\ P(g|N) &= \frac{P(N|g)P(g)}{P(N)} = \frac{0,3 * 0,7}{0,45} = 0,47 \\ P(p|N) &= 1 - P(g|N) = 0,53 \end{aligned}$$

6. We have thus the following decision tree:



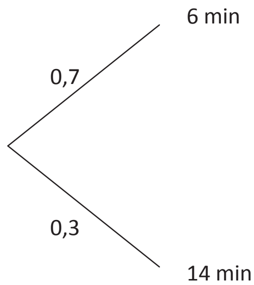
In this case:

$$E[Gain] = 0,55 * 11\ 405\ 000 + 0,45 * 641\ 000 = 915\ 725.$$

Exercise 2
Let us note PL the event "a patient is suffering from a severe pathology" and let us note Pl the event "a patient is suffering from a mild pathology". In this case, we have:

$$P(PL) = 0.3$$
$$P(Pl) = 0.7$$

a. The mean time: $E(t) = 0,7 * 6 + 0,3 * 14 = 8,4\ min$



b. Let us note DL the event "a patient who has answered at least one yes to the questionnaire" and let us note Dl "a patient who has answered no to all the questions". In this case, we have:

$$P(DL|PL) = 1 \Rightarrow P(Dl|PL) = 0$$
$$P(DL|Pl) = 0.2 \Rightarrow P(Dl|Pl) = 0,8$$

We can compute $P(DL)$ and $P(Dl)$ using the total probabilities law:

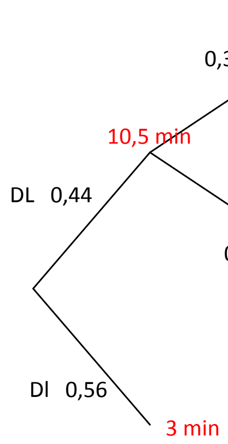
$$P(DL) = P(DL|PL)P(PL) + P(DL|Pl)P(Pl) = 1 * 0,3 + 0.2 * 0,7 = 0,44$$

$$P(Dl) = 0,56$$

We can also compute $P(PL|DL)$, $P(PL|Dl)$, $P(Pl|DL)$ and $P(Pl|Dl)$ using the Bayes formula:

$$P(PL|DL) = \frac{P(DL|PL)P(PL)}{P(DL)} = \frac{1 * 0,3}{0,44} = 0,6818$$
$$P(Pl|DL) = 1 - P(PL|DL) = 0,3182$$
$$P(PL|Dl) = \frac{P(Dl|PL)P(PL)}{P(Dl)} = 0$$
$$P(Pl|Dl) = 1$$

We have the following tree:
Mean admission time: $E(t) = 2 + 0,44 * 10,5 + 0,56 * 3 = 8,3\ min.$



c. Let us note RDL the event "a patient who has answered at least once yes to the second questionnaire" and RDl "a patient who has answered no to all the questions of the second questionnaire". We know that in this case:

$$P(RDL|PL, DL) = 1 \Rightarrow P(RDl|PL, DL) = 0 P(RDL|Pl, DL) = 0,1 \Rightarrow P(RDl|Pl, DL) = 0,9$$

In this case, we can compute $P(RDL|DL)$ and $P(RDl|DL)$ using the total probabilities law:

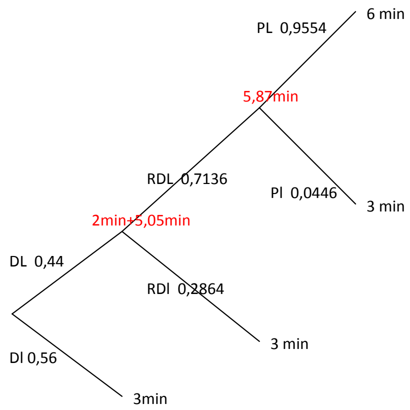
$$\begin{aligned} P(RDL|DL) &= P(RDL|DL, PL)P(PL|DL) + P(RDL|DL, Pl)P(Pl|DL) \\ &= 1 * 0,6818 + 0,1 * 0,3182 = 0,7136 \\ P(RDl|DL) &= 0,2864 \end{aligned}$$

Finally, we can compute the probabilities $P(PL|DL, RDL)$ and $P(Pl|DL, RDL)$ using the Bayes formula:

$$P(PL|DL, RDL) = \frac{P(RDL|PL, DL)P(PL|DL)}{P(RDL|DL)} = \frac{1 * 0,6818}{0,7136} = 0,9554$$

$$P(Pl|DL, RDL) = 1 - P(PL|DL, RDL) = 0,0446.$$

We have the following tree: Mean admission time: $E(t) = 2 + 0,44 * 7,05 + 0,56 * 3 = 6,78\ min$.



The best solution is thus to use two questionnaires.

Exercise 3
If we compute all the utilities, we have:

$$U(G_1) = u(50)\frac{1}{3} + u(50)\frac{1}{3} + u(50)\frac{1}{3}$$

$$U(G_2) = u(100)\frac{1}{3} + u(50)\frac{1}{3} + u(0)\frac{1}{3}$$

$$U(G_3) = u(50)\frac{1}{3} + u(0)\frac{1}{3} + u(50)\frac{1}{3}$$

$$U(G_4) = u(100)\frac{1}{3} + u(0)\frac{1}{3} + u(0)\frac{1}{3}.$$

We have then:

$$U(G_1) - U(G_2) = u(50)\frac{2}{3} - u(100)\frac{1}{3} - u(0)\frac{1}{3}$$

$$U(G_3) - U(G_4) = u(50)\frac{2}{3} - u(100)\frac{1}{3} - u(0)\frac{1}{3}.$$

So, if G_1 is prefered to G_2, G_3 is prefered to G_4.

Exercise 4
Here is a possible answer (you can find several others):

	e_1	e_2	e_3
a_1	5	0	25
a_2	10	15	5
a_3	0	45	0

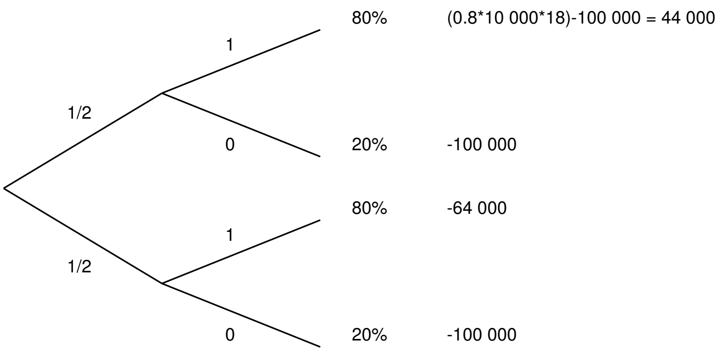
Exercise 5
Let us consider the following example:

	e_1	e_2
a_1	1	0
a_2	100	98

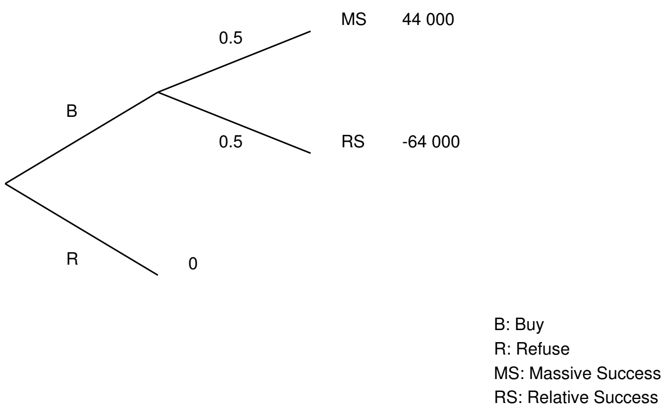
If we apply this rule, we should choose a_1 which is clearly unfavourable in comparison with a_2.

Exercise 6

1. We have the following values tree:

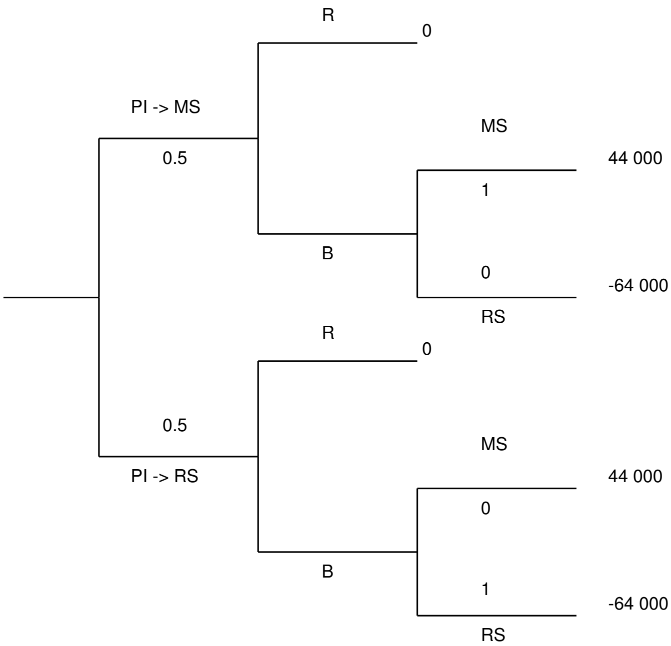


The expectation is $E = 0.5 * 44\ 000 + 0.5 * (-64\ 000) = -10\ 000$ so we won't take the risk. The decision tree can be represented as follows:



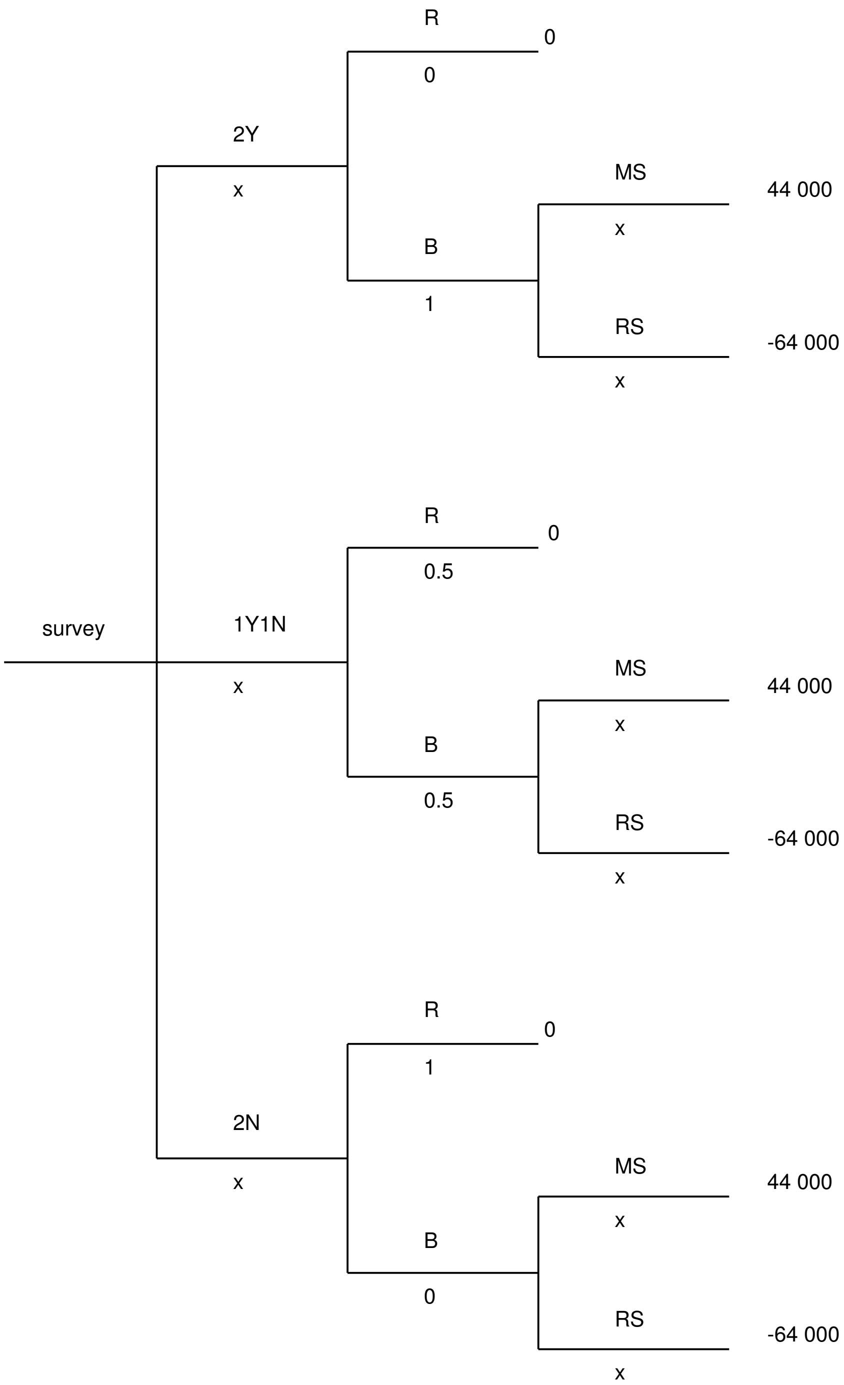
The Expected Value Without Information is thus $EVWoI = max(0; -10\ 000) = 0$

2. Let us consider that the cost of doing nothing is X, so a massive success will lead to an expectation of $44\ 000 - X$.
   We have the following decision tree with a Perfect Information (PI):



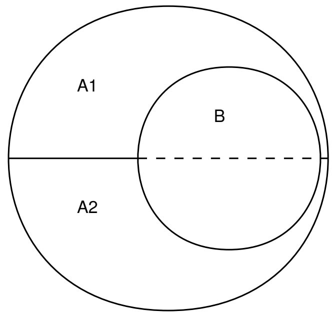
   The Expected Value With a Perfect Information is thus: $EVWiPI = 0.5*max(44\,000;0)+0.5*max(-64\ 000;0) = 22\ 000$. So the maximum we would be ready to pay for a perfect information is $EVWoI - EVWiPI = 22\ 000$.

3. We have the following decision tree with the survey (the 'x' are missing probabilities):



We thus need to compute $P(2Y)$, $P(1Y1N)$, $P(2N)$, $P(MS|2Y)$, $P(RS|2Y)$, $P(MS|1Y1N)$,

$P(RS|1Y1N)$, $P(MS|2N)$ and $P(RS|2N)$. Note that we consider the case "$1Y1N$" and "$1N1Y$" at the same time so the probability is simply multiplied by 2.
In order to compute $P(2Y)$, $P(1Y1N)$ and $P(2N)$ we can use the following reminder about the partitioning axiom:
Consider the following population situation:

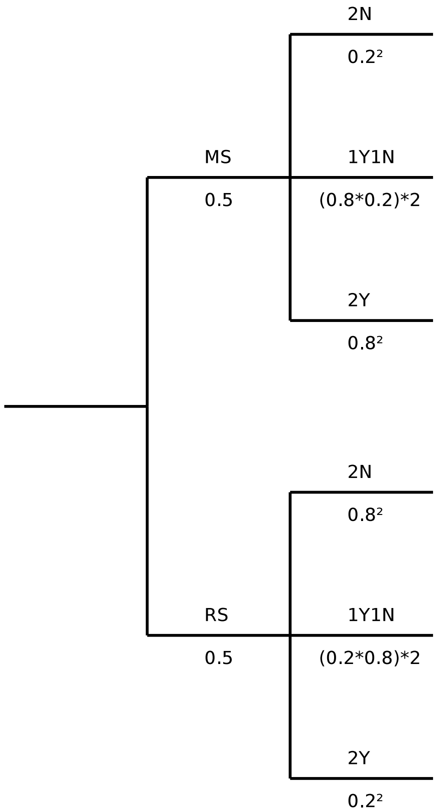


$$\begin{aligned}
P(B) &= P(B \cap (A1 \cup A2)) \\
&= P((B \cap A1) \cup (B \cap A2)) \\
&= P(B \cap A1) + P(B \cap A2) \\
&= P(B|A1)P(A1) + P(B|A2)P(A2)
\end{aligned}$$

We have thus for $P(2Y)$:

$$P(2Y) = P(2Y|MS)P(MS) + P(2Y|RS)P(RS)$$

In order to compute $P(2Y|MS)$, let us check the probability tree associated to the survey:



The probabilities can be computed as follows: we know that if it is a MS, the probability to have $1Y$ is 0.8 and $1N$ is 0.2, so $P(2Y|MS) = 0.8^2$, $P(1Y1N|MS) = (0.8 * 0.2) * 2$, and $P(2N|MS) = 0.2^2$. The same principle applies for the RS cases. So, we have:

$$\begin{aligned} P(2Y) &= P(2Y|MS)P(MS) + P(2Y|RS)P(RS) \\ &= 0.8^2 * 0.5 + 0.2^2 * 0.5 = 0.34 \end{aligned}$$

We have the same probability for $P(2N) = 0.34$, and $P(1Y1N) = 1 - 0.34 - 0.34 = 0.32$. Next, we need to find $P(MS|2Y)$:

$$\begin{aligned} P(MS|2Y) &= \frac{P(MS \cap 2Y)}{P(2Y)} \text{ (Bayes)} \\ &= \frac{P(2Y|MS)P(MS)}{P(2Y)} = \frac{0.8^2 * 0.5}{0.34} = 0.94 \text{ (Bayes again)} \end{aligned}$$

We thus have $P(RS|2Y) = 1 - 0.94 = 0.06$.
Since the case "$2N$" is the dual of the case "$2Y$", we will have $P(MS|2N) = 0.06$

and $P(RS|2N) = 0.94$.
The probabilities for $P(MS|1Y1N)$ and $P(RS|1Y1N)$ are obviously 0.5 using the same formula.
The Expected Value With a 2-persons survey is thus:

$$\begin{aligned} EVWi2 &= 0.34 * max(0; 0.94 * 44\ 000 + 0.06 * (-64\ 000)) \\ &\quad + 0.32 * max(0; 0.5 * 44\ 000 + 0.5 * (-64\ 000)) \\ &\quad + 0.34 * max(0; 0.06 * 44\ 000 + 0.94 * (-64\ 000)) \\ &= 12\ 756.8 \end{aligned}$$

So the maximum we would pay for this survey is $44\ 000 - 12\ 756.8 = 31\ 243.2$ which is more than the maximum for a perfect information!!! This is mainly due to the fact that we have to consider the cost of the information being imperfect since the people who participate to the survey can be mistaken.