
Exercise sheet n° 5 :

5-1. *Channel with additive noise.* We consider a memoryless channel with additive noise, with input X and output $Y = X + Z$, where Z is a random variable with $P(Z = 0) = P(Z = a) = 1/2$. We assume that the alphabet of the input is $\mathcal{X} = \{0, 1\}$ and that the variable Z is independent of X. Calculate the capacity of this channel as a function of the parameter a (which is a whole number and ≥ 0.).

5-2. We consider a memoryless channel with input X, which takes values $0, 1, 2$ and 3, and output $Y = X + Z \pmod 4$, where Z is a random variable which takes values $-1, 0$ and 1 with probabilities $1/4, 1/2$ and $1/4$, respectively. In addition, X and Z are independent.

(a) What is the probability distribution $p^*(x)$ that maximizes the mutual information ?

(b) Calculate the capacity of this channel.

(c) Calculate the capacity of a channel system that consists of two concatenations of the channel.

(d) Calculate the capacity of a channel system that consists of two times the channel in parallel. Compare the result with the results of (b) and (c).

5-3. We are given a noisy channel with a binary alphabet for the input and output, with transition matrix

$$p(y|x) = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \qquad x, y \in \{0, 1\}.$$

(a) Calculate the capacity and the probability distribution $p^*(x)$ that attains this maximum.

(b) Find the symmetric binary channel that gives the same capacity as the previous channel.

(c) Calculate the capacity of a channel with transition matrix

$$p(y|x) = \begin{pmatrix} 1 & 0 \\ 1-q & q \end{pmatrix}, \qquad x, y \in \{0, 1\}, q \in [0, 1] \tag{1}$$

5-4. *Suboptimal codes.* We consider again the channel of exercises 5-3. We chose a sequence of random error correction codes $(2^{nR}, n)$, for which each codeword is a sequence of n equiprobable random bits.

(a) This sequence does not attain the capacity (which was calculated previously). Why ?

(b) Determine the maximal transmission rate R for which the average error probability $P_e^{(n)}$ of the random codes tends to zero for a block length n tending to infinity.

5-5. *Pre-treatment of the output.* We consider a channel characterized by the transmition matrix $p(y|x)$ and a given capacity C. We decide to increase this capacity by introducing a "pre-treatment" of the output, $\tilde{Y} = f(Y)$. The resulting channel is thus $X \to Y \to \tilde{Y}$.

(a) Show that $\tilde{C} > C$ is impossible and that thus, this method does not work.

(b) In which case does this pre-treatment not decrease the capacity of the channel ?

Web site :

http ://quic.ulb.ac.be/teaching/