TP 3 – Computation Tree Logic

CTL Syntax Formulas of CTL are built over a set of atomic propositions $\mathcal{P}$ and satisfy the following syntax :

$$\phi \ ::= \ \top \mid \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid \exists \bigcirc \phi \mid \forall \bigcirc \phi \mid \exists \phi \ U \ \phi \mid \forall \phi \ U \ \phi$$

where $p \in \mathcal{P}$.

Exercice 1 Let K be the Kripke structure with three states $\{s_1, s_2, s_3\}$ defined as :

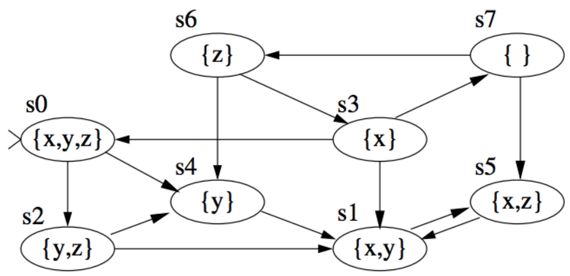
- initial state : s_1
- atomic propositions : p
- labels : $L(s_1) = L(s_3) = \{p\}$ and $L(s_2) = \varnothing$
- transitions : $s_1 \to s_1$, $s_1 \to s_2$, $s_2 \to s_3$ and $s_3 \to s_3$.

Consider the two following formulas :

1. CTL : $\forall\Diamond\forall\Box p$
2. LTL : $\Diamond\Box p$

Tell whether these formulas are satisfied or not in K.

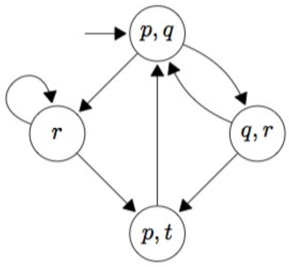
Exercice 2 Does $s_0 \models \forall\Diamond\forall\Box x$ in the following structure ?



Exercice 4

Exercice 4 For the following Kripke structure, and the following statements, replace ? by either $\models$ or $\not\models$:

1. $\mathcal{K}$? $\forall\Diamond q$
2. $\mathcal{K}$? $\forall\Box(\exists\Diamond(p \vee q))$
3. $\mathcal{K}$? $\exists\bigcirc(\exists\bigcirc r)$
4. $\mathcal{K}$? $\forall\Box\forall\Diamond q$



Exercice 5 Tell whether the following equivalences are true or false :

1. $\forall\Box\phi \;\equiv\; \phi \wedge \forall\bigcirc\forall\Box\phi$
2. $\exists\Box\phi \;\equiv\; \phi \wedge \exists\bigcirc\exists\Box\phi$
3. $\forall\Diamond\phi \;\equiv\; \phi \wedge \forall\bigcirc\forall\Diamond\phi$
4. $\exists\Diamond\phi \;\equiv\; \phi \wedge \exists\bigcirc\exists\Diamond\phi$

Exercice 6 Give a structure which satisfies $\forall\Box\exists\Diamond p$ (CTL) but not $\Box\Diamond p$ (LTL).

Exercice 7 Which of the following assertions are correct ? Prove it or give a counter-example.

(a) If $s \models \exists\Box\, a$, then $s \models \forall\Box\, a$.

(b) If $s \models \forall\Box\, a$, then $s \models \exists\Box\, a$.

(c) If $s \models \forall\Diamond\, a \vee \forall\Diamond\, b$, then $s \models \forall\Diamond\,(a \vee b)$.

(d) If $s \models \forall\Diamond\,(a \vee b)$, then $s \models \forall\Diamond\, a \vee \forall\Diamond\, b$.

Exercice 8 A CTL formula is in existential normal form ($CTL^{\exists}$) if it is of the form

$$\phi \;::=\; \top \mid p \mid \phi \wedge \phi \mid \neg\phi \mid \exists\bigcirc\phi \mid \exists(\phi\; U\; \phi) \mid \exists\Box\phi$$

Show that any CTL formula can be put in existential normal form.

Exercice 9 Give a Kripke structure $K = (I, S, R, L)$, we define the function $SAT : CTL^{\exists} \rightarrow S$ recursively as follows :

- $SAT(\top) = S$
- $SAT(a) = \{s \in S \mid a \in L(s)\}$
- $SAT(\phi_1 \wedge \phi_2) = SAT(\phi_1) \cap SAT(\phi_2)$

– $SAT(\neg\phi) = S \setminus SAT(\phi)$
– $SAT(\exists \bigcirc \phi) = \{s \in S \mid Post(s) \cap SAT(\phi) \neq \varnothing\}$
– $SAT(\exists(\phi_1\ U\ \phi_2))$ is the least fixpoint of the equation

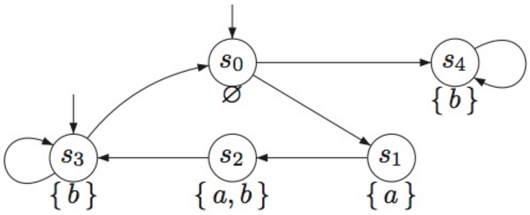
$$X = SAT(\phi_2) \cup (SAT(\phi_1) \cap Pre(X))$$

– $SAT(\exists\Box\phi)$ is the greatest fixpoint of the equation

$$X = SAT(\phi_1) \cap Pre(X)$$

Compute $SAT(\phi)$ for the following formulas and tell whether there are satisfied or not.

$$\begin{aligned}
\Phi_1 &= \forall(a\,\mathsf{U}\,b) \vee \exists \bigcirc\ (\forall\Box\, b)\\
\Phi_2 &= \forall\Box\,\forall(a\,\mathsf{U}\,b)\\
\Phi_3 &= (a \wedge b) \rightarrow \exists\Box\,\exists \bigcirc\ \forall(b\,\mathsf{W}\,a)\\
\Phi_4 &= (\forall\Box\,\exists\Diamond\,\Phi_3)
\end{aligned}$$



Exercice 10 Same questions for the following formulas and structure :

1. $\forall\Diamond(\exists(p\ U\ \exists \bigcirc t))$
2. $\exists(p\ U\ (r \wedge \exists \bigcirc (\exists \bigcirc p)))$