TP 1 – Comparing Structures

Definition 1. *Given two Kripke structures $\mathcal{K}_1 = (\mathcal{I}_1, \mathcal{S}_1, \mathcal{R}_1, \mathcal{L}_1)$ and $\mathcal{K}_2 = (\mathcal{I}_2, \mathcal{S}_2, \mathcal{R}_2, \mathcal{L}_2)$, and a relation $\sigma \subseteq \mathcal{S}_1 \times \mathcal{S}_2$, we say that σ is a simulation relation if*

1. *$(s_1, s_2) \in \sigma$ implies $\mathcal{L}_1(s_1) = \mathcal{L}_2(s_2)$.*
2. *$\forall (s_1, s_2) \in \sigma, \forall s_1' \in \mathcal{S}_1$ such that $s_1 \rightarrow_{\mathcal{K}_1} s_1'$, there exists $s_2' \in \mathcal{S}_2$ such that $(s_1', s_2') \in \sigma$ and $s_2 \rightarrow_{\mathcal{K}_2} s_2'$.*
3. *for all $s_1 \in \mathcal{I}_1$, there exists $s_2 \in \mathcal{I}_2$ such that $(s_1, s_2) \in \sigma$.*

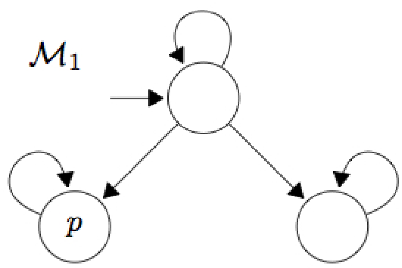
We write $\mathcal{K}_1 \preceq^S \mathcal{K}_2$ whenever there exists a simulation relation σ between $\mathcal{K}_1$ and $\mathcal{K}_2$, and say that $\mathcal{K}_2$ simulates $\mathcal{K}_1$.

Exercice 1 Show that $\preceq^S$ is transitive.

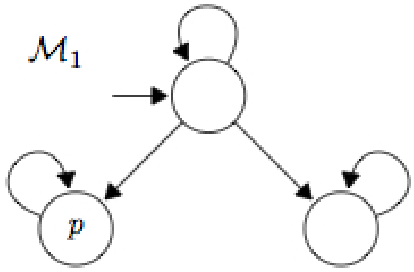
Exercice 2 The relation $\preceq^S$ between Kripke structures is a preoder. Explain why it is, in general, not an order. Give a counter-example *(we assume that two Kripke structures $\mathcal{K}_1$ and $\mathcal{K}_2$ are equal if they are isomorphic).*

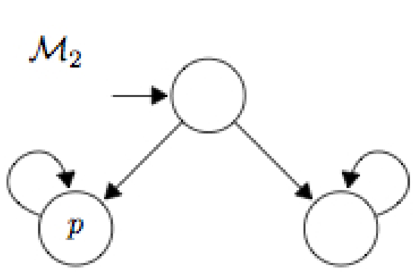
Exercice 3 Show that $\simeq^S$, defined by $\mathcal{K}_1 \simeq^2 \mathcal{K}_2$ if $\mathcal{K}_1 \preceq^S \mathcal{K}_2$ and $\mathcal{K}_2 \preceq^S \mathcal{K}_1$, is an equivalence relation.

Exercice 4 What is the language of the following Kripke structure ? (the set of variables is $\mathcal{V} = \{p, q\}$).

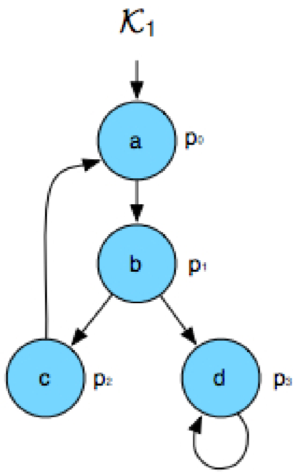


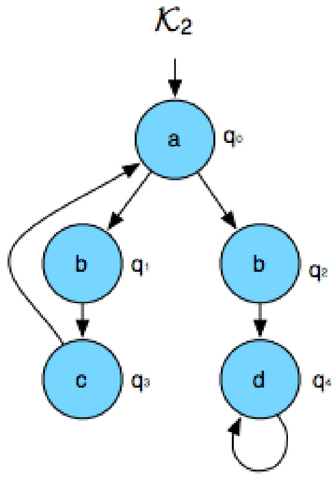
Exercice 5 Compare the following structures with $\preceq^S$:





Exercice 6 Compare the following structures with $\preceq^S$, by applying the algorithm, presented in the lecture notes, that computes the greatest simulation relation :





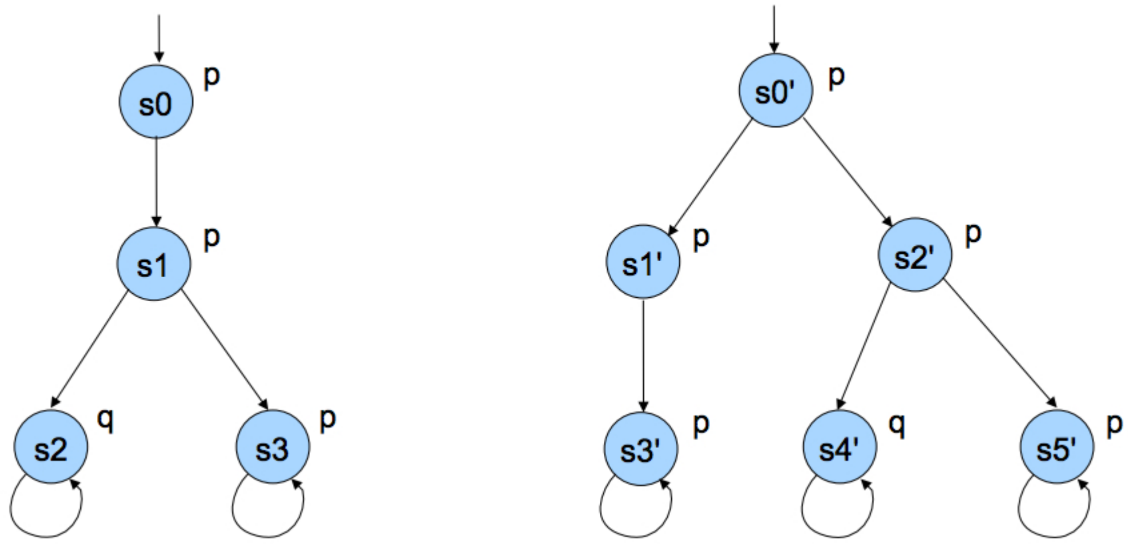
Exercice 7 Simulation implies language inclusion, i.e. $\mathcal{K}_1 \preceq^S \mathcal{K}_2$ implies $Lang(\mathcal{K}_1) \subseteq Lang(\mathcal{K}_2)$, for all Kripke structures $\mathcal{K}_1, \mathcal{K}_2$. Prove this statement and show that the converse does not hold true.

Definition 2. *Given two Kripke structures $\mathcal{K}_1 = (\mathcal{I}_1, \mathcal{S}_1, \mathcal{R}_1, \mathcal{L}_1)$ and $\mathcal{K}_2 = (\mathcal{I}_2, \mathcal{S}_2, \mathcal{R}_2, \mathcal{L}_2)$, and a relation $\sigma \subseteq \mathcal{S}_1 \times \mathcal{S}_2$, we say that σ is a bisimulation relation if*

1. *$(s_1, s_2) \in \sigma$ implies $\mathcal{L}_1(s_1) = \mathcal{L}_2(s_2)$.*
2. *$\forall (s_1, s_2) \in \sigma, \forall s_1' \in \mathcal{S}_1$ such that $s_1 \rightarrow_{\mathcal{K}_1} s_1'$, there exists $s_2' \in \mathcal{S}_2$ such that $(s_1', s_2') \in \sigma$ and $s_2 \rightarrow_{\mathcal{K}_2} s_2'$,*
3. *$\forall (s_1, s_2) \in \sigma, \forall s_2' \in \mathcal{S}_2$ such that $s_2 \rightarrow_{\mathcal{K}_2} s_2'$, there exists $s_1' \in \mathcal{S}_1$ such that $(s_1', s_2') \in \sigma$ and $s_1 \rightarrow_{\mathcal{K}_1} s_1'$,*
4. *for all $s_1 \in \mathcal{I}_1$, there exists $s_2 \in \mathcal{I}_2$ such that $(s_1, s_2) \in \sigma$,*
5. *for all $s_2 \in \mathcal{I}_2$, there exists $s_1 \in \mathcal{I}_1$ such that $(s_1, s_2) \in \sigma$.*

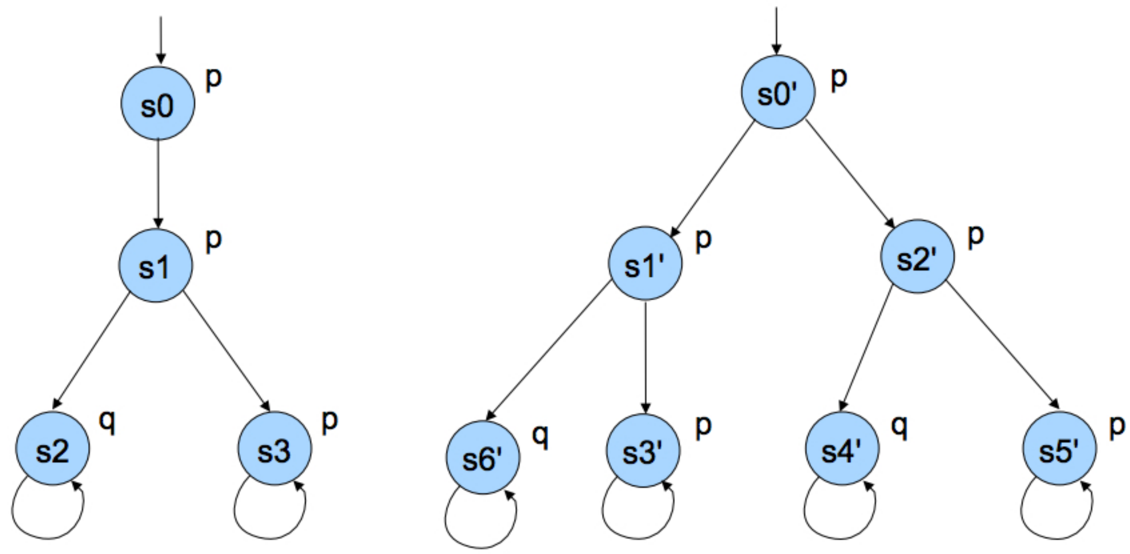
We write $\mathcal{K}_1 \simeq^B \mathcal{K}_2$ if such a relation σ exists, and say that $\mathcal{K}_1$ and $\mathcal{K}_2$ are bisimilar.

Exercice 8 Are the following two structures bisimilar ?



Give the greatest relation that satisfies conditions 1 to 3 of bisimulation relation.

Exercice 9 Same questions with the following structures $\mathcal{K}_3$ (on the left) and $\mathcal{K}_4$:



Exercice 10 Compute the quotient of $\mathcal{K}_4$ by the coarsest equivalence relation ρ compatible with its sets of transitions (use the algorithm presented in the lecture notes) to compute ρ.

Exercice 11 Prove that $\mathcal{K}_1 \simeq^B \mathcal{K}_2 \implies \mathcal{K}_1 \simeq^S \mathcal{K}_2$ and show that the converse does not hold, in general.