# INFO-F-410 Embedded Systems Design
# Game Theory

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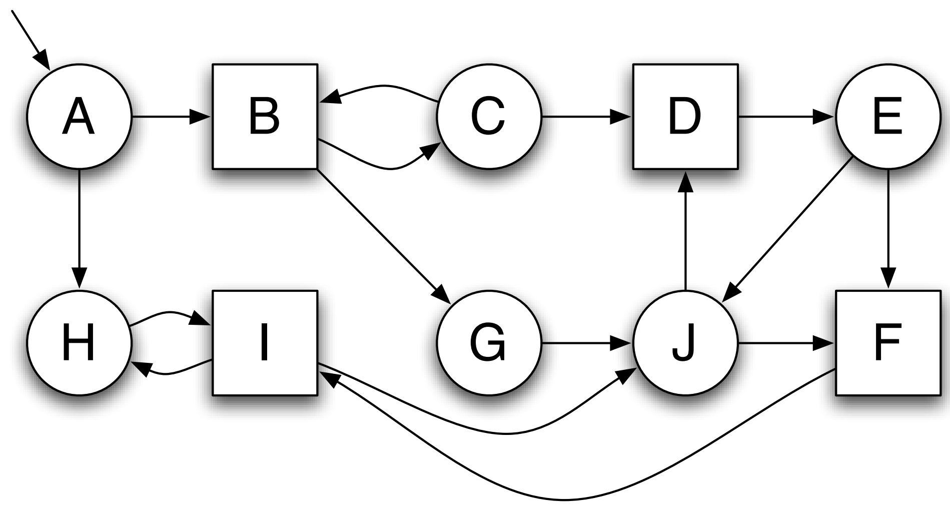
Academic year 2012–2013

## Exercise 1

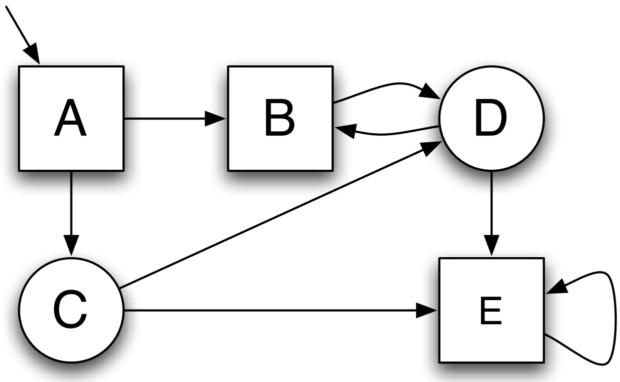
Let us consider the following arena where player $B$ plays with round nodes:




- Formalise, using Muller conditions (weak or strong), the following objectives for $B$:
  1. The play always reaches $I$.
  2. The play always reaches $I$ and $J$.
  3. The play never leaves either nodes $A$, $B$, $D$, $E$, $G$ and $J$, or nodes $A$, $B$, $C$, $D$, $E$, and $I$.
  4. The play visits at least infinitely often $J$.
  5. The play visits infinitely often exactly $A$, $I$ and $H$.
- For all these conditions try to devise a winning strategy for $B$, if you believe it exists.
- Apply the Attractor based algorithm on condition 1.

## Exercise 2

Let us consider the following arena where player $B$ plays with round nodes:

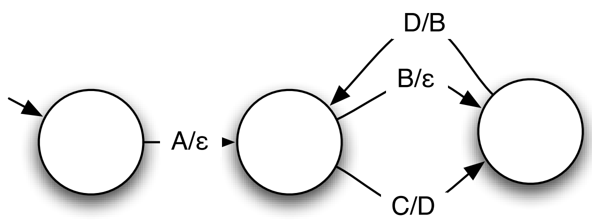



The objective for $B$ is as follows: if the play visits $C$, it can't visit $B$ but must visit $D$. If the play visits $B$, it can't visit $C$ nor $E$

- Formalise that objective as a weak Muller condition.
- To formalise strategies that need memory, one can use a Mealy machine, which is a finite automaton that produces a word on its output while recognising a word on its input. Edges in a Mealy machines are thus labelled by pairs $(i, o)$, where $i$ is the read letter and $o$ the letter produced on the output. To encode a strategy, the Meal machine should read on input the sequence of visited locations (i.e. the play), and produce on the output:

  1. Either $\varepsilon$ when the position does not belong to $B$ (the strategy we are building is for $B$ only).
  2. Or the next state to be played by $B$, according to the strategy, if the position belongs to $B$.

  Is the strategy represented hereunder winning for the objective above ? Why ?




- Apply the algorithm studied during lectures to compute a winning strategy for $B$.
- Formalise that strategy as a Mealy machine.

## Exercise 3

- In the algorithm to solve weak parity games, we have initialised the sequence of sets $A_i$ as follows:

$$\begin{aligned} A_k &= \mathrm{Attr}_A(C_k) \\ A_{k-1} &= \mathrm{Attr}_B(C_{k-1} \setminus A_k) \end{aligned}$$

  where $k$ is the maximal color and $C_i$ is the set of nodes colored by $i$. Can we replace the definition of $A_{k-1}$ by:

$$A_{k-1} = \mathrm{Attr}_B(C_{k-1}) \setminus A_k$$

  If yes, explain why. If no, give a counter-example.

- Is it correct to say that, for any set $A$ and $B$: $\mathrm{Attr}_X(A \cup B) = \mathrm{Attr}_X(A) \cup \mathrm{Atrr}_X(B)$ (for some player $X$) ? Use your answer to explain why the definition of $A_{k-2}$ given in the course is:

$$A_{k-2} = \mathrm{Attr}_A(C_{k-2} \setminus A_{k-1} \cup A_k)$$

  and not:

$$A_{k-2} = \mathrm{Attr}_A(C_{k-2} \setminus A_{k-1}) \cup A_k$$

## Answers

### Exercise 1

Objectives:

1. Weak objective: $\{S \mid I \in S\}$
2. Weak objective: $\{S \mid \{I, J\} \subseteq S\}$
3. Weak objective: $\{S \mid S \subseteq \{A, B, D, E, G, J\}$ or $S \subseteq \{A, B, C, D, E, I\}\}$
4. Strong objective: $\{S \mid J \in S\}$
5. Strong objective: $\{\{A, I, H\}\}$

Winning strategies for $B$:

1. Any strategy s.t. $A \to H$, $H \to I$
2. Any strategy s.t. $A \to B$, $C \to D$, $E \to J$, $J \to F$, $G \to J$.
3. No winning strategy. From $A$, we must go to $B$ to avoid $H$. Hence, player $A$ can bring the play to $C$. From $C$, no choice allows player $B$ to win: if player $B$ ever chooses $C \to B$, player $A$ can choose $G$ as next state, and player $B$ looses. If player $B$ ever chooses $C \to D$, player $A$ can force the game to visit $E$ and $J$ or $F$.
4. Any strategy s.t. $J \to D$, $A \to B$, $C \to D$, $E \to J$ and $G \to J$.
5. No winning strategy. Since $A$ has no input edge, it is not possible to visit $A$ infinitely often.

Attractor for $B$ of $\{I\}$:

| $i$ | $\text{Attr}^i_B(\{J\})$ |
|---|---|
| 0 | $\{I\}$ |
| 1 | $\{F, H, I\}$ |
| 2 | $\{A, E, F, H, I, J\}$ |
| 3 | $\{A, D, E, F, G, H, I, J\}$ |
| 4 | $\{A, C, D, E, F, G, H, I, J\}$ |
| 5 | $\{A, B, C, D, E, F, G, H, I, J\}$ |

Player $B$ can win from any initial position. To find a winning strategy, always go to node in a smaller attractor, i.e.:

- $A \to H$ ($\text{Attr}^2_B \to \text{Attr}^1_B$)
- $H \to I$ ($\text{Attr}^1_B \to \text{Attr}^0_B$)
- $C \to D$ ($\text{Attr}^4_B \to \text{Attr}^3_B$)
- $E \to F$ ($\text{Attr}^2_B \to \text{Attr}^1_B$)
- $G \to J$ ($\text{Attr}^3_B \to \text{Attr}^2_B$)
- $J \to F$ ($\text{Attr}^2_B \to \text{Attr}^1_B$)

## Exercise 2

Idea of the winning strategy:

- If the play visits $C$, goto $B$, then, always choose to go to $D$ from $B$.
- If the play has never visited $C$ and we are in $D$, go to $E$

Thus, choosing the right successor for $D$ requires memory.

Objective as a weak Muller:

$$\left\{ \begin{array}{c} \{C,D\},\{C,D,A\},\{C,D,E\},\{C,D,E,A\}, \\ \emptyset,\{A\},\{B\},\{D\},\{E\},\{A,B\},\{A,D\},\{A,E\},\{B,D\},\{D,E\},\{A,B,D\},\{A,D,E\} \end{array} \right\}$$

Idea of the construction: first consider all the subsets containing $C$. These must contain $D$ and can't contain $B$. Thus we are left with four possibilities (for $A$ and $E$).

Then, consider the remaining $2^4 = 16$ possibilities, and rule out the sets that contain $B$ and $E$ (since the objective says « if we visit $B$, we can't visit $E$ »).

The Mealy machine is not a winning strategy, since it always plays the same move from $D$, i.e., go to $B$. This is losing if we have visited $C$ before.

Reduction to a parity game, see Figure 1.

Computation of the winning states:

- $A_{11} = \mathrm{Attr}_A(\{17\}) = \{17, 18\}$
- $A_{10} = \mathrm{Attr}_B(\emptyset) = \emptyset$
- $A_9 = \mathrm{Attr}_A(\{11, 15, 16, 18\} \setminus \emptyset \cup A_{11}) = \{7, 11, 14, 15, 16, 17, 18\}$
- $A_8 = \mathrm{Attr}_B(\{13\} \setminus A_9 \cup \emptyset) = \{5, 6, 10, 13\}$
- $A_7 = \mathrm{Attr}_A(\{12\} \setminus A_8 \cup A_9) = \{7, 9, 11, 12, 14, \ldots, 18\}$
- $A_6 = \mathrm{Attr}_B(\{4, 7, 8, 10, 14\} \setminus A_7 \cup A_8) = \mathrm{Attr}_B(\{4, 8, 10\} \cup A_8) = \{2, \ldots 6, 8, 10, 13\}$
- $A_5 = A_7$
- $A_4 = \mathrm{Attr}_B(\{3\} \setminus A_5 \cup A_6) = \{1, 2, \ldots 6, 8, 10, 13\}$. Remark: at this point all the nodes belong either to $A_4$ or to $A_5$.
- $A_3 = A_5$
- $A_2 = A_4$
- $A_1 = A_3$
- $A_0 = A_2$.

Thus, $W_B = \{1, 2, 3, 4, 5, 6, 8, 10, 13\}$, $W_A = \{7, 8, 9, 11, 12, 14, 15, 16, 17, 18\}$. Hence $B$ has a winning strategy:

- When in $D$ and having seen $A$ and $B$ before, goto $B$ ($3 \to 4$ in the parity game).
- When in $D$ and having seen $A$, $B$ and $D$ before, goto $B$ ($8 \to 4$ in the parity game).

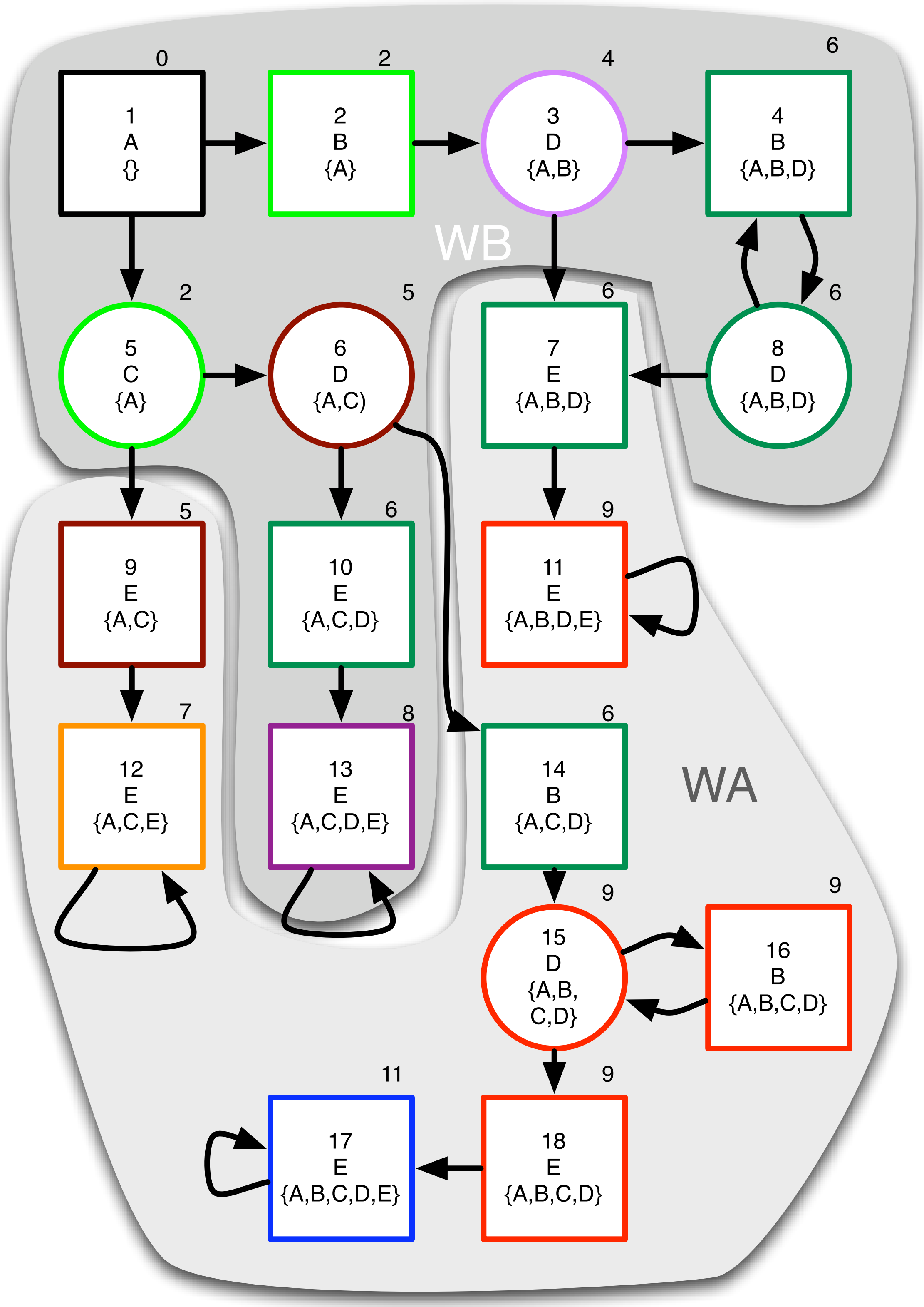
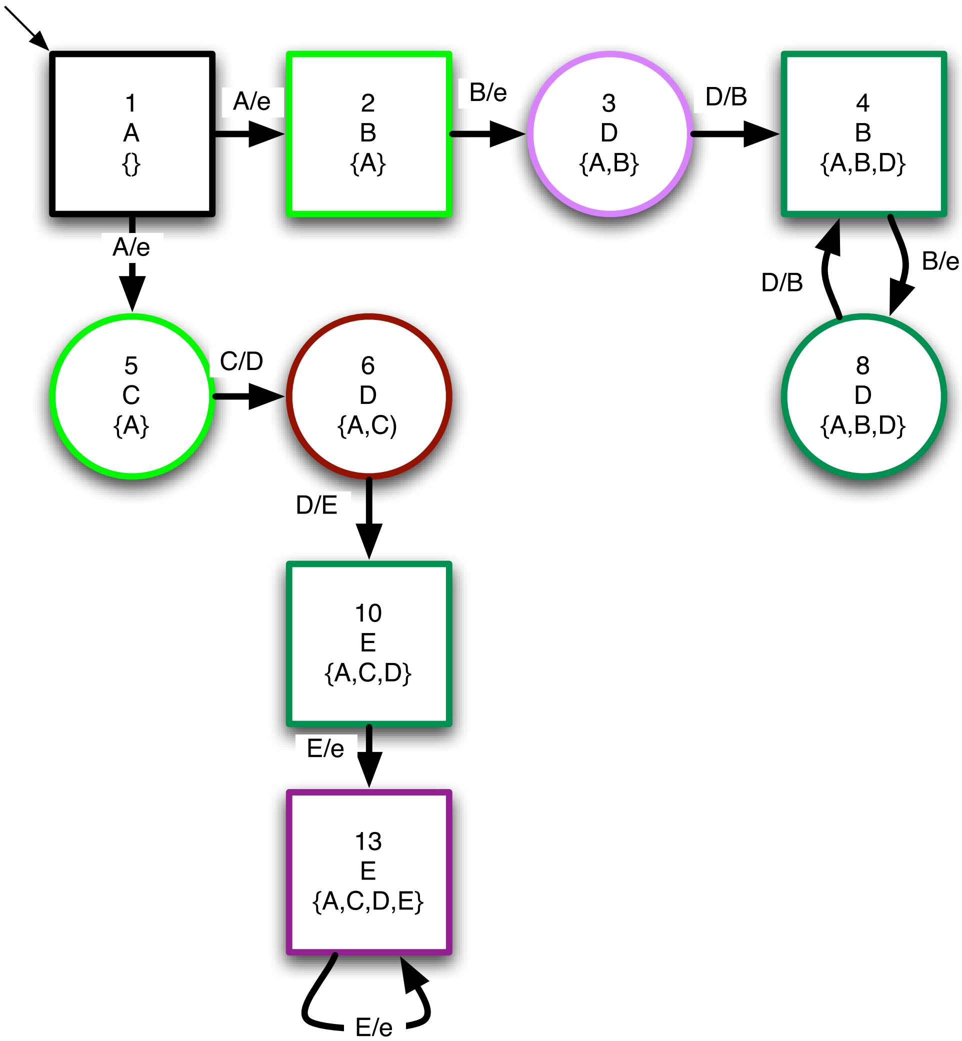



Figure 1: Reduction to a parity game, and winning regions

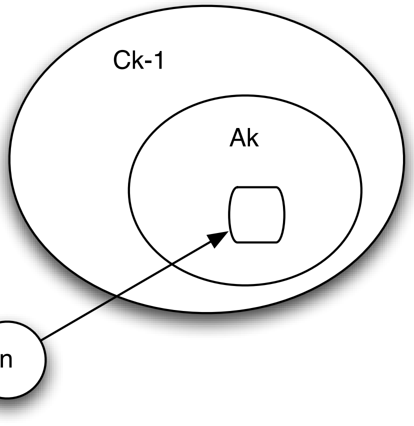
- When in $C$, an having seen $A$ before, goto $D$ ($5 \to 6$ in the parity game).
- When in $D$ and having seen $A$ and $C$ before, goto $E$ ($6 \to 10$ in the parity game).

This can be formalised as the following Mealy machine (which has to be made deterministic to be implemented. This is achieved by merging states 2 and 5):

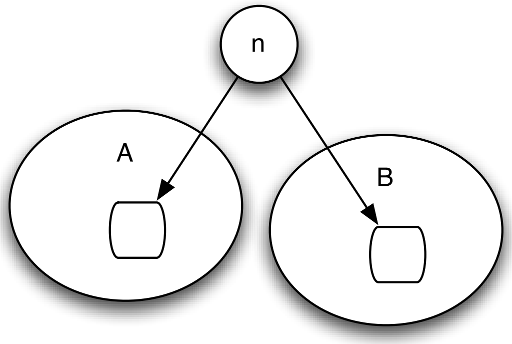



### Exercise 3

1. No, we can't, as shown on this counter-example. Clearly, $n \in \mathrm{Attr}_B(C_{k-1})$, but $n \notin A_k$, hence, $n \in \mathrm{Attr}_B(C_{k-1}) \setminus A_k$. However, $n \notin \mathrm{Attr}_B(C_{k-1} \setminus A_k)$.




2. No, this is not correct. In the counter-example hereunder, if $X$ plays with square nodes, we have $n \in \mathrm{Attr}_X(A \cup B)$, but neither $n \in \mathrm{Attr}_X(A)$ nor $n \in \mathrm{Attr}_X(B)$.




Remark that $A_k = \mathrm{Attr}_A(A_k)$. Thus, if the above property were correct, we could have:

$$\begin{aligned} A_{k-2} &= \mathrm{Attr}_A(C_{k-2} \setminus A_{k-1} \cup A_k) \\ &= \mathrm{Attr}_A(C_{k-2} \setminus A_{k-1}) \cup \mathrm{Attr}_A(A_k) \\ &= \mathrm{Attr}_A(C_{k-2} \setminus A_{k-1}) \cup A_k \end{aligned}$$

However, the second equality does not hold, as the above counter example can be applied here too.